Harriot's Calculation of the Meridional Parts as Logarithmic Tangents

JON V. PEPPER

Communicated by D. T. WHITESIDE

Summary

Many approximate solutions of the problem of constructing a true sea-chart or MERCATOR's projection were worked out in the sixteenth and seventeenth centuries. It was not until after the middle of the latter century that it became widely known that the solution was very closely related to logarithmic tangents. However, unpublished manuscripts in private possession in England show that the problem had been completely dealt with much earlier by THOMAS HARRIOT (1560—1621), the contemporary of NAPIER and BRIGGS, who is best known in the history of mathematics for his work on algebra. HARRIOT was successively scientific and navigational adviser to Sir WALTER RALEGH and under the patronage of the earl of Northumberland. As a young man he produced an approximate solution of the Mercator problem, which is now lost. He then developed theories of the conformality of stereographic projection, the rectification and quadrature of the equiangular (logarithmic) spiral, used the exponential series, devised interpolation formulae, and applied these results to the calculation of the so-called meridional parts *(latitudes croissantes)* used in the construction of a Mercator chart, which was most probably completed in 1614. Transcriptions of many of the relevant manuscripts are included in the article.

A pressing problem for navigators at the end of the sixteenth century was the construction of a true sea-chart, from which the true compass-course from one place to another could be directly read off. The solution to this question is called "Mercator's Projection", but this is misleading in two ways. First, the idea neither originated with G. MERCATOR,[1] nor was it sufficiently worked out by him, and secondly, the solution is not a true geometrical projection in the elementary sense of a projection, but a complicated mathematical transformation.

During the sixteenth century navigators used the *common* or *plane chart,* in which both latitude and longitude lines were represented by equally spaced parallel lines (Fig. 1 a). For example, to sail to a place whose latitude and longitude

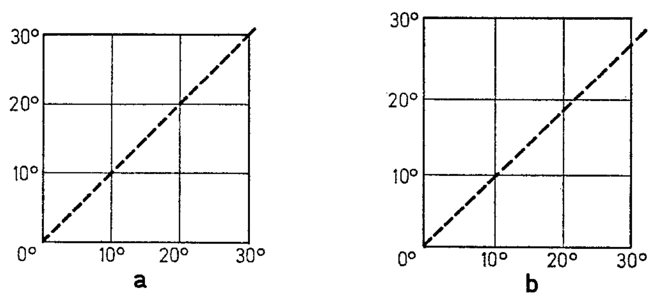


Fig. 1. (a) The plane chart, (b) the true chart

[1] *Nova et aucta orbis terrae descriptio ad usum navigantium emendate accomodata* (Duisburg, 1569).

were both 30° different from the port of departure, a course of 045° would be taken.[2] This would be quite correct if the lines of longitude on the globe were parallel, but of course they are not, and their convergence to the poles causes inaccuracies in this simple method which have serious consequences for the longer sea passages. It was seen early on that an allowance could be made for this by making the spacing of the latitude lines increase away from the equator. The question was, precisely how should this spacing increase.

There was some idea of this as early as 1511. ERHARD ETZLAUB made a pair of sundials (1511 and 1513), and there is a map on the case that is said to give a tolerable mercator projection up to about 60 degrees.[3] PEDRO NUÑEZ (1502—1578) pointed out that a course of constant compass bearing on the globe (a *rhumb line* or *loxodrome*) was not part of a great circle.[4] JOHN DEE, the friend of MERCATOR, produced tables which remained unpublished until recently,[5] but which show that he could have constructed a very tolerable true chart, probably about 1558. MERCATOR'S own map[6] came out in 1569, and shows the whole world. It was therefore not of much practical use to the mariner, who needed maps of selected smaller regions. Neither DEE'S nor MERCATOR'S actual method is known, but a reasonable conjecture will be mentioned below.

In 1594 THOMAS BLUNDEVILLE published an extract from some tables made by his friend EDWARD WRIGHT, showing how the *meridian-line* (or latitude scale) should be increased.[7] These tables were later published in greater detail by WRIGHT himself.[8] WRIGHT'S tables of 1599 are usually said to be the first tables of the so-called *meridional parts*. It has been known for some time that Sir WALTER RALEGH'S adviser, THOMAS HARRIOT (1560—1621) had similar tables in the 1580's or early 1590's.[9]

The later history of the problem in the seventeenth century is worth mentioning. SNELL and GUNTER published their tables.[10] WILLIAM OUGHTRED interested himself in the problem, but did not publish any results.[11] HENRY BOND apparently suspected the relation between the meridional parts and logarithmic tangents,[12] and JOHN COLLINS refers to this relation.[13] NEWTON looked at the question.[14]

---

[2] The winds currents and tides need their allowances, but we neglect these here.

[3] J. KENNING, "The history of geographical map projections until 1600", *Imago Mundi*, xii (1955), 17, which quotes DRECKER, *Annalen der Hydrographie* (1917).

[4] *Tratado da Esphera* (Lisbon, 1537). This includes *Tratado em defensam da carta de marear* at ff. 59—89v.

[5] E. G. R. TAYLOR, *A Regiment for the Sea* (Cambridge, 1963), 415—433.

[6] *Op. cit.* (1).

[7] *His Exercises* (London, 1594) 2nd edn. 1597.

[8] *Certaine Errors in Navigation* (London, 1599) 2nd enlarged edn. 1610.

[9] E. G. R. TAYLOR & D. H. SADLER, "The Doctrine of Nauticall Triangles Compendious", *J. Inst. Navigation*, vi (1953), 131—147.

[10] EDMUND GUNTER'S *De Sectore et Radio* (London, 1623) had been circulated in manuscript for many years before it was printed.

[11] J. COLLINS, *The Marriners Plain Scale New Plain'd* (London, 1659), 42—47.

[12] E. HALLEY, "An Easie Demonstration of the Analogy of the Logarithmick Tangents to the Meridian Line or sum of the Secants, &c.", *Phil. Trans. Roy. Soc.*, xvi (1695), 202—214.

[13] *Op. cit.* (11).

[14] D. T. WHITESIDE, *The Mathematical Papers of Isaac Newton* (Cambridge, 1967), i, 466—467, 473—475, and Plate iv.

JAMES GREGORY proved the exact relation in 1668,[15] and ISAAC BARROW wrote on the problem.[16] JOHN WALLIS produced a solution in series,[17] and EDMOND HALLEY gave a simplified solution of the problem, based on the conformality of stereographic projection, together with another solution in series.[18] But even at the end of the century, it was possible to write nonsense on the subject.[19]

It is the purpose of this article to describe HARRIOT'S unpublished work on the subject. First, the mathematical nature of the problem must be stated.

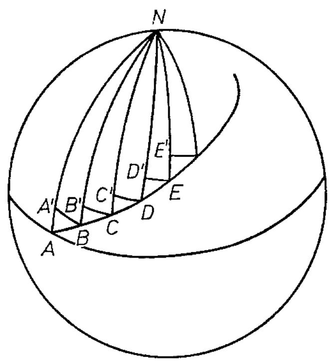


Fig. 2. Nautical triangles on the globe

In Fig. 2, $ABCDE\ldots$ is a rhumb line whose constant bearing is α. $A'B$, $B'C$, $C'D$, &c., are parts of latitude lines (small circles). We may choose $B, C, D, \ldots$ so that $AB=BC=CD=\cdots=d$, say. Now imagine that the triangles $AA'B$, $BB'C, \ldots$ are small enough for the following equations to apply (as they will in the limit):

$$AA'=BB'=CC'=\cdots=d\cos\alpha, \tag{1}$$

$$A'B=B'C=C'D=\cdots=d\sin\alpha. \tag{2}$$

AA', BB', &c. are direct measures of latitude, so that the total difference of latitude (d. lat.) is $\cos\alpha\sum d$. This result is used in modern *traverse tables*.[19a] But $A'B$, $B'C$, &c. do *not* measure difference of longitude (d. long.). They are, instead, the *departures* or distance from the meridian, and are related to the d. long. by the simple equation

$$\textit{d. long.} = \textit{departure} \times \sec \textit{latitude}. \tag{3}$$

Thus the equation

$$\text{d. long.} = \sin\alpha \sum d\sec\theta, \tag{4}$$

[15] *Exercitationes Geometricae* (London, 1668), 14—17.

[16] *Lectiones Geometricae* (London, 1670), 111.

[17] "Concerning the Collection of Secants; & the True Division of the Meridian in the Sea-Chart", *Phil. Trans. Roy. Soc.*, xv (1685), 1193—1201.

[18] *Op. cit.* (12).

[19] RICHARD NORRIS, *The manner of finding of the true Sum of the Infinite Secants of an Arch, by an infinite series* (London, 1685).

[19a] *E.g.* INMAN'S *Nautical Tables* (London, 1957), 26—100.

where θ is the latitude, is an approximation whose accuracy may be improved by choosing a smaller d, and hence by adding the secants at a closer interval of latitude. For the mercator chart, in which a rhumb line is to be represented by a straight line, and the angle of bearing shown correctly, the d. long. on a course of 45° gives the correct extension of the meridian line or meridional parts (mer-parts). Thus

$$M(\lambda) = k \lim \sum \sec \theta, \tag{5}$$

where k is a scale constant, gives the mer-part of latitude λ. The triangles $AA'B$, &c., of Fig. 2 are *nautical triangles*, and their solution is illustrated in Fig. 3.

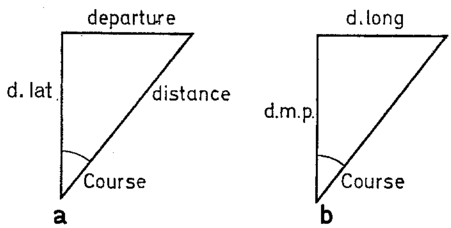


Fig. 3. Solution of the nautical triangle. (a) plane chart, (b) true chart. (*d.m.p.* is *difference in meridional parts*)

Unless d. longs. are involved the plane chart will give correct answers, *e.g.* the distance to be sailed on a given course to increase or decrease the latitude by one degree is 60 sec *Course* in nautical miles. Tables of these quantities for "raysing or laying a degree of latitude" are common in the sixteenth century nautical handbooks, and their accuracy is explained by the convenient coincidence that the plane chart gives the correct result. But the sailor hoping to increase both his latitude and longitude by equal amounts by steering a course of 45° would find his d. lat. short.[20] HARRIOT's solution of the nautical triangle is shown in Tables 2 and 3.[20a] These are taken from his incomplete tract *The Doctrine of Nauticall Triangles Compendious.*[21]

From (5) it follows that the exact result for the mer-parts is

$$M(\lambda) = k \int_0^{\lambda} \sec \theta \, d\theta = k \ln \tan\left(\tfrac{1}{4}\pi + \tfrac{1}{2}\lambda\right). \tag{6}$$

The seventeenth century constructor of tables of these mer-parts was happy adding secants, which is what WRIGHT had done (at 1′ intervals) and what HARRIOT had done in his early days;[22] even HALLEY said this was good enough for practical purposes.

That HARRIOT worked on the problem was first pointed out in 1953.[23] Those parts of the unpublished manuscript which are relevant to the present question amount to three sheets of calculations, and some queries, and these were analysed

[20] We ignore the difficult problem of the practical determination of longitude, not solved until the 18th century, prior to which dead reckoning based on traverses was used.

[20a] See *Appendix*.

[21] Leconfield 241, vi.b.

[22] *Op. cit.* (21); and perhaps MERCATOR and DEE as well.

[23] *Op. cit.* (9).

by D. H. SADLER. He concluded that HARRIOT was using the logarithmic formula (6) at the time of the manuscript (most probably 1594), before the invention of logarithms or the introduction of the calculus. However, the manuscripts published here for the first time[24] show that this statement goes too far when based on the evidence that was being considered in 1953. Fortunately, it turns out that the claim can be substantiated on the basis of other manuscripts which are much more extensive, but which have been hitherto virtually unnoticed. I refer to Leconfield 240, taken in conjunction with British Museum Add 6786, ff. 1—217, *passim*. In these papers, amounting to well over 600 half-folio sheets, HARRIOT constructs a table of logarithmic tangents, with the appropriate scale constant, that is, a table of meridional parts, by a direct method (not by a series), and to a very high degree of accuracy, far removed from the tedious but elementary addition of secants.

This article contains an account of this more advanced and mathematically highly interesting construction, based on the manuscripts in which it appears, namely:

A. Leconfield 241, vi.b (the *Doctrine*),
B. Leconfield 240,[25] which consists of about 450 sheets,
C. BM Add MS 6786, ff. 1—217,
D. BM Add MS 6789, ff. 17—18.

A has been considered elsewhere,[26] and is only necessary here for the light it throws on the fundamental equation

$$\tan\left(\tfrac{1}{4}\pi - \tfrac{1}{2}\phi_n\right) = \tan^n\left(\tfrac{1}{4}\pi - \tfrac{1}{2}\phi_1\right). \tag{7}$$

B and *C* should be taken together; they were probably separated early in the 1620's, and well before the separation of 1810.[27] Taken together they form a substantially complete set of calculations, even including some of the routine arithmetic. *D* deals with stereographic projection.[28]

First, it is necessary to give a brief account of the mathematical parts of *A*.[29] By taking a basic latitude of 10°, and using the relation[30]

$$\tan(45° - \tfrac{1}{2}\phi_n) = \tan^n(45° - \tfrac{1}{2}\cdot 10°)$$

latitudes are obtained whose mer-parts are $10n$ degrees ($n = 1, 2, \ldots, 5$). The values are

$$10°,\ 19°\,42',\ 28°\,51',\ 37°\,15'\,50'',\ 44°\,50'\,20''.$$

[24] In the *Appendix*.

[25] These are the contents of the "black box full of papers of rhumbs" catalogued after HARRIOT's death, at BM Add MS 6789, f. 449.

[26] *Op. cit.* (9), and my *Harriot's work on Mathematical Navigation* (London *M. Sc.*, 1967), Ch. iv.

[27] When the 3rd earl of Egremont, the descendant of HARRIOT's second patron, HENRY PERCY, 9th earl of Northumberland, gave most of HARRIOT's non-astronomical manuscripts to the British Museum, where they are at Add MSS 6782—6789. Copies of *A* exist at Harley 6002, ff. 35—42, and of a few sheets of *B* at Harley 6001, ff. 19—24.

[28] The diagram for *D* is reproduced by J. A. LOHNE, "Thomas Harriot als Mathematiker", *Centaurus*, xi (1965), 25. The same article prints a small extract from the mer-parts tables of *B*.

[29] These are at pp. 13—14, 19 and 21 of the MS, and are reproduced here in the *Appendix*. They include some latin text. The remainder of the *Doctrine* is in English.

[30] The genesis of this equation will shortly appear.

The mer-parts of these latitudes are then multiples of that of 10°, which HARRIOT takes to be 603′,088,252.[31] He repeats the calculation for a basic latitude of 19°, getting 19° and 36° 4′ 128/3661; and for 1°, getting 1° and 1° 59′ 5517/5623, and a mer-part of 119′,02449 for 1° 59′ by interpolation, which is 42 more than the value to be obtained by summation of secants from minute to minute. In the last case the basic latitude is taken to be 1′, with a mer-part of 1′,000,000,0. Repeated squaring leads, for $n=64$, to 1° 3′ 5523/5713, and extrapolation gives 64′,033,263,2 as the mer-part of 64′. The actual value is 64′,003,697. The error is explained by the erroneous starting value of 9,997,093 for tan 44° 59′ 30″, obtained (perhaps) by linear interpolation, which has been raised to a high power, with attendant truncation errors. HARRIOT knew that his value for the mer-part was not correct, as he puts 037 under it. His value for 1°, on the same page, is 60′,0031,225; the correct value is 60′,00305.

It is likely that CLAVIUS' tables were used, as HARRIOT'S incorrect value (9,826,974 for 9,826,972,6) for tan 44° 30′ agrees with that given there.[32] In doing this work HARRIOT had not used any method resembling the integral calculus, as the work on stereographic projection which he did, which is about to be described, shows; nor is he using logarithms, except in the sense that anyone who constructs them uses them, for example NAPIER or BRIGGS. HARRIOT'S methods could not be the same as theirs, for reasons that will be seen. As with NAPIER, the idea of a base does not appear, and it is an interesting coincidence that the mer-parts are directly related to the *Differentiae* column of NAPIER'S tables.[33]

---

[31] It should be 603′,069,579; his calculations of 1613—14, whose description is about to be given, obtained 603′,069,583,6 (Leconfield 240, f. 311).

[32] C. CLAVIUS, *Theodosii Tripolitae Sphaericorum Libri III* (Rome, 1586), 219. There is a note "Theodosius de sphaera graecolat" at Leconfield 241, viii, 36v., but this is much later, on the back of a sunspot observation for 26 June 1612, in a list of books including *Opus Palatinum* (Neustadt, 1596), the great RHETICUS-OTHO trigonometric tables.

[33] *Mirifici Logarithmorum Canonis Descriptio* (Edinburgh, 1614). The text was translated by EDWARD WRIGHT for the English edition *A description of the Admirable table of logarithmes ... with an addition ... by Henry Briggs* (London, 1616). Two small points may be dealt with here. SADLER (*op. cit.* (9)) mentions a *Conjectarium* on p. 13 of the *Doctrine*. This should be *Consectarium*, an inference or conclusion. There are one or two other slight mistranscriptions, such as *subtensa* for *subtrahe*. Secondly, the incorrect value for tan 44° 59′ 30″ seems to date from a time before HARRIOT'S work on differences and interpolation given at BM Add MS 6782, ff. 107—144 (ff. 145—253 also contain much related material). This is the *De Numeris Triangularibus et inde De Progressionibus Arithmeticis Magisteria magna T. H.*, which would have enabled him to use the general forward difference formula, and thus avoid the error in the 7th figure. It is this treatise, which is copied at Harley 6083, ff. 403—444, and referred to there at f. 404 as HARRIOT'S "booke of triangular numbers" that is referred to by J. A. LOHNE, "The Fair Fame of Thomas Harriot", *Centaurus*, viii (1963), 69—84, when he says (at p. 81) "I can prove that Harriot's treatise was studied by Sir Charles Cavendish, by Sir Thomas Aylesbury and also by John Pell". TAYLOR and SADLER also draw attention to the rather elementary arithmetic used in HARRIOT'S demonstrations and examples, but this hardly seems worth special comment. Indeed, modern navigational methods based on, say, the *Sight Reduction Tables for Air Navigation* (London, 1967) 3rd edn., need no more than the same elementary arithmetic for their use, and this is an advantage for it reduces the accidental errors associated with more involved methods.

The major gap in the method of the *Doctrine* is that it does not show how to construct a *useful* table of mer-parts. Instead, it solves the much easier inverse problem of finding the ratio of latitudes for which the corresponding mer-parts are simple multiples. Thus at Leconfield 241, vi.b, 14 there is (in translation) "Having doubled, note the complement. Which number is the latitude, of which the length to be found is double the first, of the limit, that is, of the 45° rhumb". But what is needed for practical purposes is to have the mer-part tabulated for given latitudes. Taken with the uncertainty of a basic latitude, that is, the question of a latitude scale, it is seen that HARRIOT was at the time of the *Doctrine* far from turning his ingenious use of equation (7) above, which, it will be suggested below, he got by stereographic projection, into the definite tables that he produced later. It is a curiosity of historiography that SADLER's conclusions on the *Doctrine*, mentioned earlier, turn out to be correct, not on the evidence available to him, but on that about to be presented, which has a much later date.

Table 3 showed that the *Doctrine* did solve the nautical triangle, on the assumption that mer-parts tables are available and have reasonable accuracy, but it was presumably left incomplete because there was not yet an answer to the problems mentioned in the last paragraph. HARRIOT was evidently not satisfied with the tables formed by the addition of secants, or he could have circulated the *Doctrine* or even published it, as HUES had led readers of his *Tractatus* to hope.[34]

We can now turn to HARRIOT's full calculations, which appear to date from about 1613—1614. There is a single sheet[35] of calculations with the date "Jan: 23. 1613/1614", *i.e.* 23 January 1614. This is a check sheet, not in HARRIOT's hand, but (most likely) that of his assistant, CHRISTOPHER TOOKE.[36] It forms one of a series in which two identical calculations are carried out, in the two hands, and in which the concurrence of results is noted. The actual calculations form part of the extensive arithmetical drudgery that was necessary, and the whole work must have taken place over a period of at least several months.[37] I see no difficulty in supposing that the work to be described dates from 1613 to 1614, because even though the beginnings of the method used may have

[34] ROBERT HUES, *Tractatus de Globis et eorum usu* (London, 1594), 111, at the end of his (rather vague) description of the use of rhumb lines on a globe, looks forward to the publication of the work of HARRIOT on the subject. Such a work was not published, but HUES, who was in the RALEGH-NORTHUMBERLAND circle, may have seen the *Doctrine*.

[35] BM Add MS 6786, f. 99.

[36] Several check sheets, ff. 99—101 and elsewhere are ticked and initialled "Ch" (= Chris) or "fact". TOOKE received generous legacies from HARRIOT in 1621, HENRY STEVENS, *Thomas Hariot &c.* (London, 1900), 196—198. STEVENS' book is the standard work on HARRIOT, but needs to be brought up to date in some respects.

[37] So much for the unsupported remark that "After 1610, his health ... collapsed entirely. He did little more until his death from cancer in 1621", R. H. KARGON, "Thomas Harriot, the Northumberland Circle and early Atomism in England", *J. History of Ideas*, xxvii (1966), 128—136. Many of the astronomical observations were after 1610, there is the work about to be described, that on the Comet of 1618 (Leconfield, 241, vii), and on the Calendar (*id.*, i), dated 1616, and, as evidence of his continued wide reading, there is the bookseller's bill (*id.*, iv, 9—9v.) of 31 October 1619, which contains over 40 items, enough to keep anyone busy.

originated before 1600, most of the essential details remained to be worked out; and even when this was done (which may have been before 1613) the actual computations remained to be done.[38]

It is necessary to separate the description of the construction into several parts, as the whole of the calculation is rather complex. It will therefore be considered under the following headings:

(i) the conformal property of stereographic projection,

(ii) the fundamental formula obtained from (i),

(iii) the theory of the equiangular (logarithmic) spiral,

(iv) the derivation of a fundamental constant, from (iii), and of certain fundamental tables, or tables of powers of this constant, to form the basis of

(v) the direct calculation of the mer-parts of selected latitudes, including special devices for high latitudes,

(vi) the determination of intermediate values, based on

(a) *corrections* to linear interpolation, and

(b) certain *adjustments* required to bring the table into order from time to time, and finally

(vii) a description of the tables of d. longs. on the seven rhumbs, and a few isolated problems.[39]

(i) The Conformality Theorem

A reproduction of HARRIOT's own proof of the conformality property of stereographic projection is given in the *Appendix*.[40] The basic argument is shown

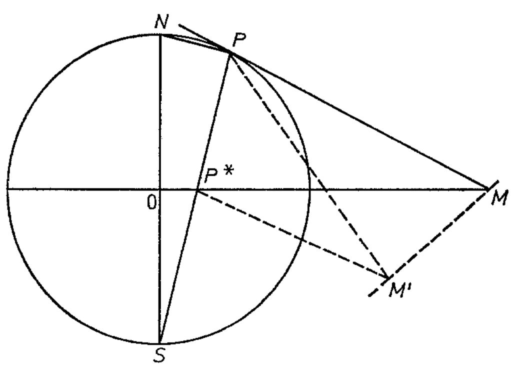


Fig. 4. Proof of the conformality of Stereographic Projection

[38] CHRISTOPHER HILL's statement that HARRIOT had "apparently solved some of the most complicated problems in navigational mathematics a generation before Napier, Briggs, and Gunter", *The Intellectual Origins of the English Revolution* (Oxford, 1965), 140, can be traced back to TAYLOR & SADLER's article, *op. cit.* (9), and while it does not apply to the work being described, is true of his other navigational work, see *op. cit.* (26), Chapters iii and iv.

[39] The 7 rhumbs, *viz.* courses on the various points of the compass, $11\frac{1}{4}°$, $22\frac{1}{2}°$, &c. DEE and WRIGHT had produced such tables earlier.

[40] Transcribed from *D*, with a translation.

in the following simplified proof. In Fig. 4, NPS is a meridian section of the sphere, and M the point in that section where the tangent plane at P meets the equatorial plane of the sphere. M' is a point on the line of intersection of these two planes. The projection of P on the equatorial plane is P^*. Consider the two triangles PMM' and P^*MM'. Then $P\hat{M}M' = P^*\hat{M}M' = \frac{1}{2}\pi$. Also, $PM = P^*M$, as $S\hat{P}M = S\hat{N}P = \pi - P\hat{P^*}O = P\hat{P^*}M$. MM' is common to both triangles, hence they are congruent, and so $M\hat{P}M' = M\hat{P^*}M'$, which completes the proof.[41]

Conformality may provide a link between HARRIOT's theorem on stereographic projection, and his theorem on the area of a spherical triangle. It is remarkable that no-one had obtained the latter result before his time,[42] but such appears to be the case.[43]

It is to be noted that in the mapping of a spherical surface onto a plane surface, a spherical triangle cannot be made exactly similar to a plane triangle. For example, the angles in the former will exceed π, the constant total of those in the latter. The limiting equality of these angle totals is a necessary condition for a particular mapping to be conformal. It is not clear if HARRIOT had some such connection in mind, but it is a possibility, and it would be difficult to find another link, as astronomy, the traditional application of spherical trigonometry, had never needed the angle sum result.[44]

(ii) The Fundamental Formula

HARRIOT's derivation of the fundamental formula (7) from the conformal property does not appear to have survived, but the argument is elementary, and although the following is not necessarily the exact reasoning used, being

---

[41] An alternative, instant, proof is obtained by observing that the ratio of the lengths of line segments at P^* and P is SP^*/SP, by similar triangles, and that this is independent of the direction of the element. This also shows that the local scale at P^* is SP^*/SP.

[42] Or perhaps either result, see (44) below.

[43] GEORGE HAKEWILL's monument, *An Apologie or Declaration of the Power and Providence of God in the Government of the World* (London, 1630) 2nd edn. (a massive contribution for the moderns against the ancients) quotes a letter which BRIGGS wrote to HAKEWILL, picking out HARRIOT's determination of the area of the spherical triangle as an especial contribution of a modern to an old problem (pp. 301—302 of the 3rd edn. of 1635). Some of HARRIOT's work on this theorem is considered by J. A. LOHNE, *op. cit.* (28). There is a copy of HARRIOT's rule for the area at Harley 6083, f. 323, with a date including 18 September 1603.

[44] The area theorem was published by ALBERT GIRARD, *Invention nouvelle en Algebre* (Amsterdam, 1629), sig. G1v—H4v, and also by B. CAVALIERI, *Directorium generale uraniometricum* (Bologna, 1632), 316. Dr. D. T. WHITESIDE assures me that (to his knowledge) no one before HARRIOT went one whit beyond PTOLEMY's *Planispherium* (Venice, 1588), Prop. 19, or, before HARRIOT, had any inkling that stereographic projection preserves angles. Nor do CLAVIUS, *In Sphaeram Ioannis de Sacra Bosco* (Rome, 1582) or *Astrolabium* (Rome, 1593), nor FRANÇOIS D'AIGUILLON, *Opticorum Libri Sex* (Antwerp, 1613) mention any curve other than a circle. So it seems that not only is the fundamental formula (proved in the next section) original, but so is the remarkable conformality theorem whereby it was obtained. The conformality property was well-known in the 1670's in England, to COLLINS and HOOKE, for example. (I am grateful to Dr. WHITESIDE for this information, and the references.)

expressed in modern notation, it is related to some of the numerical work,[45] and also shows that the result is an easy consequence.

A rhumb line, which cuts meridian lines at a constant angle, is projected stereographically into an equiangular spiral, which cuts its radius vectors at the same constant angle. If $r_0, r_1, \ldots, r_n$ are the lengths of equally spaced radius vectors to the spiral, then

$$r_1/r_0 = r_2/r_1 = \cdots = r_n/r_{n-1}, \tag{8}$$

and hence

$$r_n/r_0 = (r_1/r_0)^n. \tag{9}$$

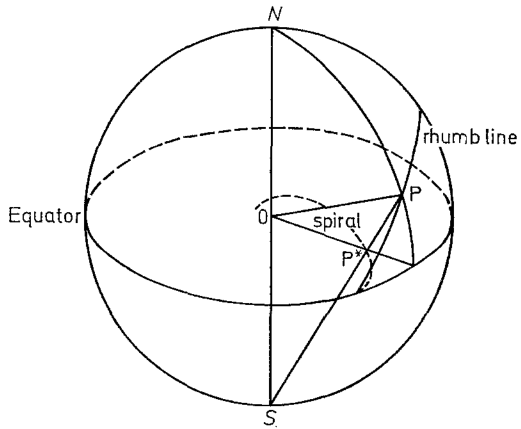


Fig. 5. Proof of the fundamental formula

In Fig. 5, *PNOS* is a plane section of a globe of radius r_0, poles N and S, centre O. P is a point on the surface through which the rhumb passes, P^* its projection on the equatorial plane. ϕ_n is the latitude of P, and $OP^* = r_n$. Then

$$r_n/r_0 = OP^*/OS = \tan O\hat{S}P^*, \quad \text{where} \quad O\hat{S}P^* = \tfrac{1}{2} N\hat{O}P = \tfrac{1}{2}(\tfrac{1}{2}\pi - \phi_n).$$

Hence

$$r_n/r_0 = \tan(\tfrac{1}{4}\pi - \tfrac{1}{2}\phi_n). \tag{10}$$

Equations (9) and (10) lead to the result used in the *Doctrine*, namely equation (7), where ϕ_1 is a basic latitude, to be chosen. We shall see how the problem of scale is dealt with, and how the formula may be applied to the construction of tables.[45*]

(iii) The Equiangular Spiral

Although HARRIOT's work on the equiangular spiral is related to the needs of the work on mer-parts, it goes further. What may have started as an auxiliary investigation grew into an independent enquiry, which would by itself rank high in the history of the rectification and quadrature of curved lines. The achievement was to obtain general results for both the length and area of an equiangular

[45] Leconfield 240, ff. 431, 439—440.

[45*] The logarithmic nature of the structure is shown by the exponentiation in formula (7).

spiral, and in a form leading to constructible lines and areas.[46] HARRIOT also obtained a very good approximation to what we would write as $e^{-\theta}$ for $\theta = 1'$, but care must be taken in the interpretation of this.

Before describing HARRIOT's work on this topic, I will set out, in modern notation, how an investigation from first principles could obtain his results. This will not only show the problems involved more clearly, but may throw light on gaps that appear in his papers. It is necessary to assume that the spiral is the limit of an approximating curve formed by straight line segments. In Fig. 6, $O\hat{a}b = \alpha$, the angle of the spiral. HARRIOT is mainly interested in the case $\alpha = \frac{1}{4}\pi$, but he does also touch on $\alpha = \frac{1}{8}\pi$, and the argument can be presented quite generally. $O\hat{b}a = \frac{1}{2}\pi$, and d_1 is the mid-point of ad_0, d_2 of ad_1, &c., and d_n of ad_{n-1}.

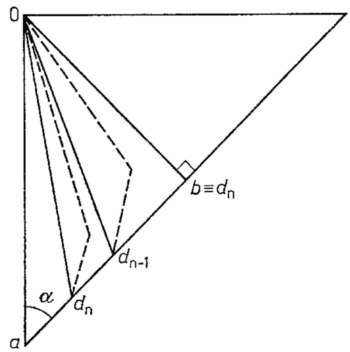


Fig. 6. Approximations to the spiral

The dotted lines represent triangles similar to Oad_n, Oad_{n-1}, and the sides ad_n and the corresponding sides of the similar triangles represent the line segments approximating the spiral. Let Oa be the unit of length. Then

$$bd_n = (1 - 2^{-n}) \cos \alpha,$$

and, writing d_n for Od_n, PYTHAGORAS' theorem gives

$$d_n^2 = \sin^2\alpha + (1 - 2^{-n})^2 \cos^2\alpha,$$

and hence

$$d_n^2 = 1 - (2^{1-n} - 2^{-2n}) \cos^2\alpha. \tag{11}$$

By similar triangles the arc length of the approximating curves will be

$$s_n = (ad_n)(1 + d_n + d_n^2 + \cdots) = (ad_n)/(1 - d_n),$$

that is,

$$s_n = \cos\alpha/2^n(1 - d_n) = \sec\alpha(1 + d_n)/(2 - 2^{-n}), \tag{12}$$

[46] This is distinguishable from the theoretical, but non-constructible, results for the circumference and area of a circle, known since antiquity, in particular to ARCHIMEDES, who obtained the well-known inequality $3\frac{10}{71} < \pi < 3\frac{1}{7}$. HARRIOT's rectification and quadrature is the first example of such a construction for any curve. As late as 1637 DESCARTES was to deny the possibility of such constructions (*Discours de la Methode* (Leyden, 1637), 340—341). DESCARTES' error was made public by EVANGELISTA TORRICELLI (1608—1647), *Opere* (Faenza, 1919), ii, 349—399; TORRICELLI's work was also on the equiangular spiral, and is completed by a proposition (found by AGOSTINI in 1928) which appears in the *Opere*, iv (1944), 297—299. The whole work may be seen in E. CARRUCCIO, *de infinitis spiralibus* (Pisa, 1955).

from (11). Hence, as $d_n \to 1$,

$$s = \lim s_n = \sec\alpha. \tag{13}$$

Similarly, the total area under the approximating curve is

$$\begin{aligned} A_n &= \tfrac{1}{2}(ad_n)\sin\alpha/(1-d_n^2) \\ &= \sin\alpha\cos\alpha/2^{n+1}(1-d_n^2) \\ &= \tan\alpha/(2^2-2^{1-n}), \end{aligned} \tag{14}$$

again from (11). Hence

$$A = \lim A_n = \tfrac{1}{4}\tan\alpha. \tag{15}$$

This method involves the double use of limiting processes. First the area or length of each approximating curve is obtained as the limit of a geometric series, and secondly this curve (formed by straight line segments) is supposed to tend to the curved spiral line.[47]

HARRIOT's papers *de Helicis* are mainly at Leconfield 240, ff. 211—253.[48] In the first folio (f. 211) he takes the crudely approximating curve shown in the diagram at the top of that folio (see Fig. 7), and obtains, first, the sum of the radii vectores, which is

$$2b + \sqrt{2bb},$$

where b is the length ab, from the sum of the series

$$b + \sqrt{bb/2} + \sqrt{bb/4} + \sqrt{bb/8} + \cdots = bb/(b - \sqrt{bb/2}) = 2b + \sqrt{2bb}.$$ [49]

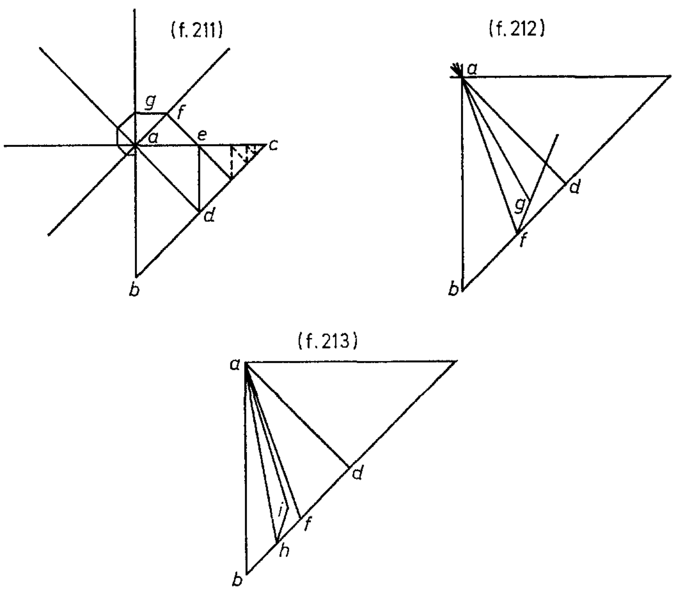


Fig. 7. Approximations to the spiral

[47] HARRIOT was interested in atomism, and his papers *de infinitis* (BM Add MS 6782, ff. 362—375) seem to indicate that he was a mathematical atomist, as well as a physical one. Perhaps he thought of the spiral as being in some sense made up of line segments. But the evidence is slight, and the actual results show the concrete achievement.

[48] See the *Appendix*.

[49] This is not quite the form of notation he uses.

His use of the sum of an infinite geometric series is to be noted in so far as he draws no special attention to it. It must be assumed that he had obtained the result in his earlier work, as it was far from being common knowledge, or an easy idea, even at a later date.[50] Next, HARRIOT finds the sum of the arc lengths to be $10+\sqrt{200}$ if $b=10$, and finally the area of the sum of the similar triangles is

$$bb/4+bb/8+bb/16+\cdots=(bbbb/16)/(bb/4-bb/8)=bb/2.$$

On f. 212, the approximating line segments are based on bf $(=\frac{1}{2}\,bd)$ and the corresponding three results obtained; on f. 213 this is done for $bh=\frac{1}{2}\,bf$ (see Fig. 7). Taking $b=10$ the results can be summarized in Table 1.

Table 1. *The approximating curves*

	Sum of radii vectores	*Sum of line segments*	*Sum of areas*
bd	$20+\sqrt{200}$	$10+\sqrt{200}$	50
bf	$(80+20\sqrt{10})/3$	$(20\sqrt{2}+10\sqrt{5})/3$	$33\frac{1}{3}$
bh	$(320+200\sqrt{2})/7$	$(40\sqrt{2}+50)/7$	$28\frac{4}{7}$

The consequences for the areas are shown at f. 214 *(Rationes triangulorum)*, where HARRIOT takes $\frac{1}{4}Oa^2$ as his unit, and shows that the areas (the A_n of above) form a sequence 2/1, 4/3, 8/7, &c., tending to 1. This gives $A=\lim A_n=\frac{1}{4}$, corresponding to equation (15) where $\alpha=\frac{1}{4}\pi$.

There is no such direct consideration for the (more difficult) case of the sequence s_n, but we can infer that he knew the result $s=\sec\alpha$ from (in particular) ff. 221 and 224;[51] and f. 227 *(De longitudine Helices 2$\overline{2}$. 30')*. The very difficult case of the sum of the radii vectores is briefly considered on f. 229.[52] On f. 217 an approximation to the length of the radius vector after one complete turn of the spiral of 45° is obtained. The angle at a is taken to be 1', so that (Fig. 8) if $\widehat{abf}=45°$ (an approximation later considerably improved), then

$$\begin{aligned} af=ae-fe=ae-be\tan fbe &= 10^{10}(\cos 1'-\sin 1'\tan 44°\,59') \\ &= 9{,}997{,}092{,}387. \end{aligned}$$

This is raised to the 675th power, corresponding to 11° 15', and it is interesting to note that HARRIOT consciously uses the binary decomposition of 675.[53] Now

[50] For example, GREGORY of St. Vincent, *Opus Geometricum* (Antwerp, 1647), 865—954, considers the matter at great length (I thank Dr. WHITESIDE for this reference). Some of HARRIOT's other work is at BM Add 6782, f. 368 *(Ratio Achilles)* and Add 6783, f. 124, which is a geometrical construction "The first & second terme of an infinite progression decreasing being geuen: to find the poynt geometrically where the progression endeth".

[51] See *Appendix*.

[52] The temptation to regard this as an integration of the exponential function should be resisted.

[53] See F. V. MORLEY, "Thomas Harriot (1560—1621)", *The Scientific Monthly* (New York), xiv (1922), 60—66; and J. W. SHIRLEY, "Binary Numeration before Leibniz", *American Journal of Physics*, xix (1951), 452—454, which refers to BM Add MS 6786, f. 346v—347.

$675 = 512 + 128 + 32 + 2 + 1$, and these powers, obtained by repeated squaring, are multiplied to give the 675^{th} power. This in turn is repeatedly squared to give

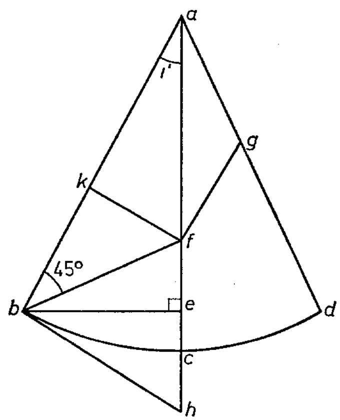


Fig. 8. (adapted from f. 224)

the radius vector for 360°, which turns out to be 18,708,345. The angles of which these powers are the tangents are noted beside the numbers.[54]

f. 221 contains important work in which a numerical approximation to the total tength of the 45° spiral and the length of its first turn are obtained. The length bf is obtained (f. 224) from the diagram (Fig. 8), as 4,112,584, and the infinite geometric series $bf + fg + \cdots$ is summed, in which the common ratio is $r = af/ab$. In modern notation, but using HARRIOT's lettering,

$$\begin{aligned}\omega &= bf + fg + \cdots\\ &= bf + r \cdot bf + \cdots r^n \cdot bf, \quad \text{where} \quad n = 360 \times 60,\\ &= bf(1 - r^{n+1})/(1 - r)\\ &= (bf \cdot bf - bf \cdot bf \cdot r^{n+1})/(bf - bf \cdot r)\\ &= (\alpha\alpha - \beta\chi)/(\alpha - \beta),\end{aligned}$$

which is the result used. Here $\alpha = bf = 4{,}112{,}584$; $\beta = fg = 4{,}111{,}388$; $\chi = zy = 7{,}693$; so that $\alpha - \beta = 1196$ and $\omega = 14{,}115$,aaa,aaa. The length of the first turn is thus 14,115,aaa,aaa. On the next line we read that "Tota Helix" is 14,142,135,623, that is, precisely $\sqrt{2}$ or sec 45°. The sum of the infinite geometric series is found on f. 224, by ignoring the term $\beta\chi$, to be 14,141,5aa,aaa, that is, rather less than its true value. HARRIOT knows this, for he says "Accurata quantitas Helices curva 14,142,135,623" (*i.e.*, sec 45°). He notes that this result obtained is sec 44° 59′ 51″. The slight discrepancy is explained by

(i) a slight error for $163'$ $(= 1' + 2' + 32' + 128')$ in the original exponentiation at f. 217,

[54] Although many of the numbers are written without decimal points or an equivalent notation, I hope that they will be clear enough in what follows if I quote them directly.

(ii) the crude approximation of the triangle abf,

(iii) the ill-conditioned division by $4{,}112{,}584 - 4{,}111{,}388 = 1196$ (although this is recalculated as 1195,7805 at f. 224), and

(iv) the usual truncation errors (rounding is very uncommon in HARRIOT'S arithmetical work).

On f. 225 *(De area inclusa pera Helica. 45g)* the area is numerically approximated in a similar way to be 25,007,aaa,aaa,aaa,aaa,aaa whereas it should be 25,000,000,000,000,000,000. That more general results than these of the 45° spiral were known is shown by similar calculations at f. 227 and f. 229 where the accurate arc length for the spiral of 22° 30′, namely sec 22° 30′ = 10,823,922,003 is approximated by 10,823,aaa,aaa; and the accurate area

$$\tfrac{1}{4}\tan 22^\circ 30' = 10{,}355{,}339{,}060{,}000{,}000{,}000$$

is again nearly obtained, as 10,359,544,aaa,aaa,aaa,aaa.[55]

That HARRIOT definitely knew the area of the 45° spiral we may be sure from f. 214, already mentioned. The exactness of his other results can be inferred from the calculations of ff. 221, 224, 225, 227, and 229. I do not think that these are attempts to obtain results from which the exact formulae were to be inferred. The exact results are stated. In his later calculations for the mer-parts he uses the much better approximating triangle abf not with $a\widehat{b}f = 45°$, but with bf such that $bf/fe = \sqrt{2}$. This may not be conclusive of a rigorous proof, but it is the next best thing. It is not uncommon for a mathematician to have inadequate proofs of his pioneering results.

At the foot of f. 229, the sum of all the radii vectores, at 1′ intervals, for the spiral of 22° 30′, is obtained as the sum of an infinite geometric progression. In this case (from f. 227) the common ratio is $af/ab = 9{,}992{,}982{,}689$ and the sum ω is 14,250,472,aaa,aaa. It is noted that this is the tangent of 89° 57′ 40″ (and also nearly the secant of the same angle), and the comment "sed qua*e*re" added. The similar calculations for the spiral of 45° appears at the foot of f. 225, where ω = 34,392,47a,aaa,aaa. The secant of 89° 59′, namely 34,377,468,aaa,aaa is noted, with the following comment:

> fortasse iste numeros est sum*ma* proportionaliu*m* in Helice insistentiu*m* quaru*m* anguli sunt vnius minuti in amplitudine. nam tales proportionales sint paulo breviores quam alterae.[56]

On f. 241 we find the result that the arc length from $r = 1$ is given by

$$s = (1 - r)\sec\alpha. \tag{16}$$

This is a simple consequence of the spiral's "lack of memory", and appears in the manuscript in the equation $bo = bg$ on the third line of working.[57] The results at the foot of f. 221 may be translated thus:

> If the rays falling on a spiral increase or decrease uniformly, then the segments of the helix thus defined are equal.
>
> *Corollary*. It follows that the segments of a spiral, which are between equally spaced parallels, are equal.

[55] See *Appendix*.

[56] Perhaps that number is the sum of the successive proportionals in the helix, spaced at angles of 1′. Thus such proportionals may be somewhat less than above.

The *Theorem* is rather obscure, but the diagram and figures next to it[57] seem to show that it is saying that, if there are two sequences of quantities, increasing or decreasing uniformly, and two pairs are in continued proportion, then so are the others. But the two results quoted above are trivial consequences of equation (16). The following is a rough translation of the *Conclusion* or *Consectarium* of Leconfield 240, vi.b., 13:

If any section of a rhumb is made through lines from the centre of a circle, by equal angles, then the lines from the centre are in continued proportion.

This makes the calculation easier in the example above. Square 8390996. The square is 7040881*3872016*. Divide by 10,000,000 or, what is the same thing, reject the last 7 figures, and the number is 7040881, which is the same as the remainder in the fourth (column). With which number work as shown there.

2. Arcs of concentric circles, drawn from the end of their lines of proportion (measuring equal angles) are also in continued proportion, and in the same ratio.

3. Next, the sections of rhumbs subtending these angles are in the same ratio.[58]

If for *rhumb* in the above we understand its stereographic projection, then these results may be illustrated by Fig. 9, and give the following equations if $A\hat{O}B = B\hat{O}C = C\hat{O}D = \cdots$:

1. $OA:OB = OB:OC = OC:OD = \cdots = r$, say;
2. $BB':CC' = CC':DD' = \cdots = r$;
3. $AB:BC = BC:CD = \cdots = r$.

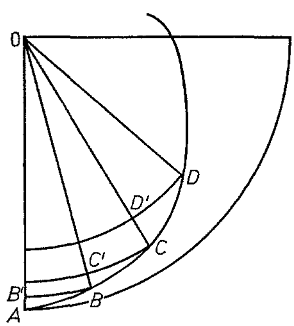


Fig. 9. Ratios for the spiral

With the interpretation just suggested, these are correct results, and thus it appears that some work on the arc length of the spiral had been done by HARRIOT in the 1590's.

In conclusion, then, HARRIOT had the general result for the length of an equiangular spiral; the area of the complete curve; and numerical estimates of certain radii vectores. These results were, as far as is known, original to him, and pre-date by a generation or more the first published work on the subject.[59]

[57] See *Appendix*.

[58] The numerals printed in italics in this extract are actually underlined in the manuscript. This is an early form of notation for showing the decimal point.

[59] In Harley 6001, f. 20v. (among the copies of Leconfield 240, f. 419, 213 and 214) there is this note: This of Mr. Hariots agrees with that ⟨doctrine⟩ of the geometricall spirall sent from Florence to Pere Mersenne; but ⟨not⟩ the demonstration not sent.

Harley 6083, f. 338, has part of a letter on the spiral, dated *Florentia mense Julio 1646*.

(iv) The Fundamental Constant

In Fig. 10, bf is an arc of a spiral of 45°, and $\widehat{baf}=1'$. The *fundamental constant* is the ratio af/ab and will be denoted by β. In fact, of course, $\beta=e^{-\theta}$, where $\theta=1'$, but HARRIOT's determination of β must proceed rather differently. His work appears mainly at ff. 432 and 434 (Fig. 10).

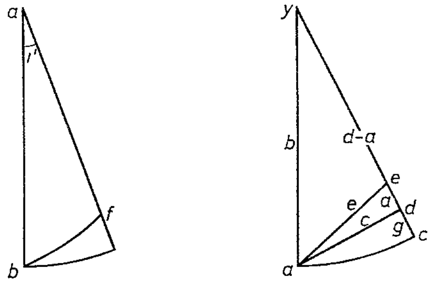


Fig. 10. Evaluation of the fundamental constant

HARRIOT's usual sign for equality is similar to the modern sign, *viz.* =, but he has two small vertical lines placed between the longer horizontal lines, at their mid-point, and the horizontal lines are sometimes found in a lengthened form. The sign for continued proportion is a large Z in which the numbers in proportion are placed at the four corners of the Z. Again, there are sometimes two vertical lines placed between the horizontal lines, and at their mid-point. The abbreviation for *tangent* is a small "○" with a short tangent line touching the "○" at the top. For *secant*, he writes ψ, and for *sine* a curly line rather like a ψ without the upright stroke, presumably in imitation of part of the sinusoidal curve.*

At f. 434,[60] ae is a straight line segment approximating the 45° spiral. If ae were part of the spiral, then from the arc length result, and its corollary, equation (16), it would follow that

$$\text{segment } ae/\text{segment } ec = \sec\alpha = \sqrt{2}.$$

This is the result written as

$$\frac{be}{b-d+a} = f, \qquad ff = 2bb.$$

The algebra then produces, on taking $c^2=e^2-a^2=b^2-d^2$ and $g=b-d$,

$$\sqrt{,3bb+dd-4bd,}-2g = a \tag{17}$$

which is correct, and this formula is used on f. 432 to obtain $d-a$, and hence β.[61] A comparison is then made between the ratios of the length of the approximating straight line segment, namely 14,142,135,623,*729*, and the actual value for the exact spiral, which is $\sqrt{2}=$ 14,142,135,623,*730,950,488,0*; there is an error of about 2 in 10^{14}. In Fig. 10, $\widehat{yae}$ is not exactly 45°.

* Strictly, these are R times tangent, &c., where R is the radius or *sinus totus*.
[60] See Fig. 10 and in *Appendix*.
[61] The commas in equation (17) are equivalent to modern brackets.

HARRIOT's value for β is 9,997,091,540,972,577,8 and as this number is the basis of all his succeeding calculations its accuracy is now considered. He himself did not, apparently, make any other checks than of the type just described.

The exact value of β is given by $r=e^{-\theta}$, that is,

$$1-\theta+\theta^2/2!-\theta^3/3!+\theta^4/4!-\cdots,$$

for $\theta=1'$. Putting $\widehat{ayc}=\theta$ in Fig. 10, so that, if $b=1$, then $d=\cos\theta$, $g=1-\cos\theta$, and $\beta=d-a$, the formula for a, equation (17), is equivalent to

$$\begin{aligned}\beta &= (2-\cos\theta)-\sqrt{(3-\cos\theta)(1-\cos\theta)}\\ &=1-\theta+\theta^2/2-\theta^3/12-\theta^4/24+\cdots.\end{aligned}$$

The discrepancy is $\beta-r$, which is approximately $\theta^3/12$. As $\theta=1'=2\pi/360\times 60=0.00029088\ldots$, this error is about $0.0^{11}205$. This may seem a trivial enough amount, but as β is eventually raised to powers of up to about 30,000, and as there are truncation errors, the calculations for very high latitudes are based on figures that are incorrect in the 7^{th} or 8^{th} decimal place. This will be seen to be the most likely explanation of the errors which occur in the 5^{th} and 6^{th} decimal places of the final tables. From the practical point of view the error is of no importance; from the theoretical side it is hardly serious, provided that its effect on the final results is anticipated.[62]

This is the place to mention HARRIOT's work on exponential series. At BM Add MS 6782, f. 67, he considered "Interest vpon interest for 7 yeares"[63]. What he does is to expand

$$(b+1/n)^{7n}/b^{7n-2}$$

as

$$b^2+7b+49(1-1/7n)/2+343(1-1/7n)(1-2/7n)/6b+\cdots$$

and then, if $b=10$, take the limit as $n\to\infty$, namely,

$$100+70/1+49/2+343/60+\cdots,$$

which, expressed in *l.s.d.*, turns out to be 207 *l.* 7*s.* 6*d.* This is a plain use of the exponential series. How it is to be dated relative to the mer-parts work is not known, but the question that arises is whether a similar method could have been used to find β.

First, note that in the above example, the instantaneous rate of growth is known; what is being evaluated is

$$100(1+1/10n)^{7n},$$

so that the (rather usurous)[64] rate is 10 per cent. But it is just this rate that is being sought when β is being found. However, it is not impossible to find β in this way. The argument would be something like this:

---

[62] This would be part of the modern *error analysis*. Note that if $\widehat{abf}$ is exactly 45° then $af/ab=1-\theta+3\theta^2/2-\cdots$, and the error is in the term in θ^2; it is about 0.0^79, and would have led to very poor final results.

[63] See *Appendix*.

[64] This was the standard rate in England until 1624, when it came down to 8 per cent.

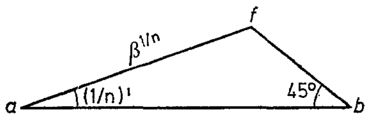


Fig. 11. The fundamental constant as an exponential limit

In Fig. 11, $\widehat{baf} = (1/n)'$, and $\sin(45° + (1/n)') = (1/\sqrt{2})/\beta^{1/n}$, so that

$$\cos(1/n)' + \sin(1/n)' = 1/\beta^{1/n}.$$

HARRIOT knew enough about the behaviour of $\sin\theta$ for small θ to have been able[65] to get from this the relation

$$1/\beta = \lim(1 + \theta/n)^n,$$

where θ is the radian measure of $1'$, and we have just seen that he could deal with this sort of thing. It is not known what, if anything, he could have made of $\beta = \lim(1 + \theta/n)^{-n}$, but that is not necessary. Thus β *could* have been obtained as an exponential limit, but it happens that it was not, and, because the slight error inherent in the method he actually used could not be estimated, this can be regretted.

Before turning to the construction of the *fundamental tables*, their necessity and use must be explained. The mer-parts are

$$k\int_0^\lambda \sec\theta\, d\theta,$$

where λ is the latitude, and k a scale constant. Expressed in the form

$$k \ln \tan(\tfrac{1}{4}\pi + \tfrac{1}{2}\lambda),$$

we have $k = 360 \times 60/2\pi = 3437.7467708$. The classic method of constructing logarithms to a given base, as used by BRIGGS,[66] is first to form a table of roots (*i.e.* powers) of that number, and then use a process of division.[67]

HARRIOT's method cannot be exactly the same, because there is no idea of a base. He does not form root extraction from a fundamental constant, like BRIGGS, but forms tables of powers, like NAPIER.[68] These are powers of β, and

[65] *Cf.* Leconfield 240, f. 239, where he compares arc lengths and chords for angles as small as 1^{v}, finding that they agree to 18 figures.

[66] *Arithmetica Logarithmica* (London, 1624) Preface.

[67] For example, for the base 10, $10^{\frac{1}{2}} = 3.16228$, $10^{\frac{1}{4}} = 1.77828$, ..., $10^{1/4096} = 1.00056$, $10^{1/8192} = 1.00028$. BRIGGS repeated his square root operations to a much larger number of decimal places, and stopped when he had two consecutive roots of the form $1 + 2e$ and $1 + e$, to the accuracy required. If the logarithm of 2 is required, the list is searched for the largest power that does not exceed 2, here 1.77828, and then 2/1.77828 is calculated. A similar search is made with this number, and the process continued until the final ratio is as near to unity as is desired. The logarithm is the sum of the indices, here beginning with $\frac{1}{4} + \frac{1}{32} + \frac{1}{64} + \cdots$ to give 0.3010. This is not precisely the process used by BRIGGS, who has more than one method. BRIGGS' own account may be seen in the introduction to his *Arithmetica Logarithmica* (London, 1624), or in the historical introduction to C. HUTTON, *Mathematical Tables &c.* (London, 1785), 61—75.

[68] *Op. cit.* (33).

form what I call his *fundamental tables.* Similar divisions and so on to those needed by BRIGGS are then done. They are rather complicated, and will be described in the next section, with selected examples.

Leconfield 240, ff. 297—300 contain the *fundamental table for minutes,* that is, β^i for $i=-1, 0, 1, 2, \ldots, 61$, complete with first second and third differences. The first differences are used in some later work, as we shall see; and the third differences, which decrease very slowly, provide a check on the figures of the main column. The multiplications for these tables do not appear to have survived, but there is no reason to suppose that they were not done directly. The table gives β^i to 13 figures, and check calculations show that this is where the values were truncated.

The value $\alpha=\beta^{60}$ is now taken as the basis for the *fundamental tables for degrees.*[69] These are tables of α^i for $i=0, 1, 2, \ldots, 131$, and then for 12 specially selected values of i up to 506.[70] No differences are given.[71] These tables were calculated by direct multiplications; this work is not in the Leconfield MSS, but in the British Museum.[72]

Much more interesting than either the *minutes* or *degrees tables* are the *fundamental tables for hundredths of a minute.* Leconfield 240, f. 281, shows an extract of the minutes table giving $\beta^{-1}, \beta^0, \beta^1, \beta^2$, together with first and second differences. The latter is 846,0 and the third difference is zero. The remainder of the sheet gives an evaluation of $\beta^{0.01}$, which I write as γ, from what is in effect the interpolation formula

$$f_q = f_0 + q\Delta_0 + \tfrac{1}{2}q(q-1)\Delta_0^2, \quad \text{where} \quad q = 0.01. \tag{18}$$

The second difference Δ_0^2 is written as e, $n=1/q=100$, and p is f_0-f_q. This, taking account of the positioning of the significant figures, leads to the equation

$$\frac{2pn+en-enn}{2} \;\text{≖}\; 2{,}908{,}459{,}1$$

and hence to

$$p \;\text{≖}\; 29{,}088{,}7{,}787.$$

ff. 277—280 determine f_n and V_n for $n=0.1$ (0.1) 1.0. First,

$$V_n = -(p+e-en), \tag{19}$$

and second,

$$f_n = \tfrac{1}{2}(2p^2+2p+enn-2pn-en).^{73} \tag{20}$$

[69] Leconfield 240, ff. 301—303.

[70] These isolated values correspond to certain high latitudes, as will be seen.

[71] Except for a very few at the top of f. 301.

[72] Add MS 6786, ff. 117—148. Leconfield 240 does give a few main calculations. For example (f. 304) it has the values of α^i for $i=10, 20, 30, \ldots, 180$. Many of these entries are double ones, the two parts differing only in the 13th or 14th figure out of 18. Further, α^{10} is found as $\alpha^5\alpha^5$, and then $\alpha^{20}=\alpha^{10}\alpha^{10}$, and so on. The BM papers give calculations like $\alpha^{110+i}=\alpha^i\alpha^{110}$ (f. 129—129v. give $i=1$ to 6). Most of this work is checked by simple repetition, in another hand. Notes like "Tried by *T*." and "Proved also by Tom" (Leconfield 240, ff. 424, 426) in addition to the checks already mentioned show that HARRIOT was aided in this work by his assistant, CHRISTOPHER TOOKE. See (36).

[73] Here, p^2 does *not* mean pp. The index is acting as a subscript, and the p^2 is in fact $f_0-f_{0.01}$. γ is $1-p$.

These values are used as checks on the *fundamental tables* of ff. 272—276, which can be (and certainly were) built up by simple addition from the constant second differences. This completes the *table*, which consists of γ^i ($i = 1$ to 100). A direct calculation will verify that

$$\gamma^{100} = (0.99999709112213)^{100} = 0.99970915409 = \beta.$$

(v) *Direct Calculation of the Meridional Parts*

HARRIOT's general plan is to calculate directly the mer-parts for 0° (1°) 80° and 0° 1′ (1°) 80° 1′; then at 30′ intervals to 83° 0′; 15′ intervals to 87° 0′; and then at 1′ intervals to 89° 59′. For the intermediate values, at intervals of 1′, between 0° and 87°, a form of interpolation, based on the addition of secants, is used. I say "based on", as the direct addition of secants is amended by two processes, which I call

(i) the *corrections*, and

(ii) the *adjustments*.

The former are rather loosely related to second differences, and the latter are made by a comparison with the directly calculated values.

These directly calculated values occupy ff. 322—399 of Leconfield 240, which is complete in this respect. The same method is not used throughout, as certain problems have to be overcome for high latitudes, but the following is a transcription of a typical calculation (f. 353).

latitudo. $\overline{20}$. 0′.	compl. $\overline{70}$. 0′.		Dimid:	$3\bar{5}$. 0′
	Ʊ	7,002,075,382		
Merid: prox:	$\overline{20}$.	7,053,446,831′		α^{20} (f. 301)
antecedens.		7,002,075,382″		Ʊ
		10,000,000,000‴		
		9,927,140,156⁗		Ʊ/α^{20}
		9,927,541,733′	25′	β^{25} (f. 298)
		9,927,140,156″		Ʊ/α^{20}
		10,000,000,000‴		
		9,999,595,492⁗		(Ʊ/α^{20})/β^{25}
		9,999,621,852	13	γ^{13} (f. 272)

difference	26,360.	9,062,4
first diff. (f. 272)	29,087.	10,000,0

The mer-part of 20° is thus 20° 25′,1390624, or 1225′,1390624. This is entered in his table (f. 41) as $\overline{20}$.25′,139062. The International Hydrographic Bureau tables[74] give 1225′,13905 which indicates an error of about 1 in 10^9 in HARRIOT's value.

What has the above calculation done? Remembering that Ʊ is the notation for tangent, I have added the explanation in the right-hand column and at the bottom left. The general argument is as follows. The *fundamental tables* are a

[74] *Tables of Meridional Parts to 5 places of decimals &c.* (Monaco, 1928), 76.

sequence of numbers

$$\gamma^1, \gamma^2, \ldots, \gamma^{100} = \beta^1, \beta^2, \ldots, \beta^{60} = \alpha^1, \alpha^2, \ldots, \alpha^{506},$$

where $\beta = e^{-1'}$, so that $\alpha = e^{-1^\circ}$ and $\gamma = e^{-0.01'}$.

(i) If the latitude is λ, the third line of the calculation determines a number a such that

$$\alpha^{a+1} < \tan(45^\circ - \tfrac{1}{2}\lambda) < \alpha^a,$$

that is,

$$(a+1) > \ln\tan(45^\circ + \tfrac{1}{2}\lambda) > a, \tag{21}$$

so that a is the number of whole degrees in the mer-parts, $M(\lambda)$. In the example given, $a = 20$.

(ii) Then the number b is found, such that

$$\beta^{b+1} < \mho/\alpha^a < \beta^b,$$

where $\mho = \tan(45^\circ - \tfrac{1}{2}\lambda)$. This is equivalent to

$$a^\circ (b+1)' > M(\lambda) > a^\circ b', \tag{22}$$

or

$$60a + b + 1 > M(\lambda) > 60a + b, \quad \text{in minutes.}$$

In the example, $b = 25$.

(iii) Next, the number c is found such that

$$\gamma^{c+1} < (\mho/\alpha^a)/\beta^b < \gamma^c,$$

so that

$$a^\circ \big(b + (c+1)/100\big)' > M(\lambda) > a^\circ (b + c/100)'. \tag{23}$$

(iv) Finally, the remaining decimals are obtained by linear interpolation, using the appropriate first difference from the *γ-table*.[75]

This is the method that HARRIOT uses, but he explains none of it, so that it is not possible to be able to know how he arrived at the complex but natural procedure just described. The tangents are to 10 places, and could equally be from the tables of RHETICUS-OTTO or PITISCUS.[76] The method above could be used throughout the construction of the tables, but $\tan\theta$ loses a significant figure between $5^\circ\,42'$ and $5^\circ\,43'$; another between $0^\circ\,34'$ and $0^\circ\,35'$; and yet another between $0^\circ\,3'$ and $0^\circ\,4'$. This effects the accuracy of the calculations first at $90^\circ - 2\times 5^\circ\,43'$, that is, between 78° and 79°; then between $88^\circ\,45'$ and 89°; and finally at about $89^\circ\,53'$. This problem is dealt with in an ingenious way. Consider f. 322:

Lat: $8\bar{0}.\ 0'$.	Comp: $1\bar{0}.\ 0'$.	Dimid: $\bar{5}.\ 0'$
	$\mho$ 874,886,635	
Merid: prox: antecedens		884,995,209′
$13\bar{8}.\ 55'$,704,523,7,16		874,886,635″
	[&c. &c.]	

[75] The *fundamental table for hundredths of a minute*. A convenient abbreviation.
[76] *Opus Palatinum* (Neustadt, 1596) and *Trigonometria* (Frankfurt, 1612) 3rd edn.

Where does this m.p.a. (merdianus proxime antecedens) and the corresponding first entry come from? Reference to f. 301 (the α-*table*) gives α^7 as 8,849,952,096,814. Hence 884,995,209 is the entry for $7+k$ degrees, where $\alpha^k = 1{,}000{,}000{,}000{,}000$. This particular m.p.a. is calculated at f. 441 as follows:

Merid: 13$\bar{1}$		1,016,335,782,358,8′
Merid. sequens		1,000,000,000,000,0″
Quaeritur angulij drā		1,000,000,000,000,0‴
	55′. &c.	9,839,267,861,642,2⁗
	55′.0′.)	9,841,284,504,602.′
		9,839,267,861,642.″
		10,000,000,000,000.‴
	0′, 70 &c.	9,997,950,833,594.⁗
	0′, 70	9,997,963,989,8
		″13,156,3.‴ — 45237,3
		′29,082,8.′ — 100000,0
	‴	10,000,0″
	⁗	4,523 7,3⁗
Ergo angulus drā		55′,70,452,37,3
Ergo pro:		1,000,000,000,000
erit		13$\bar{1}$. 55′,704,523,3.

It follows that the m.p.a. for $\tan\theta = 0.01$ will be $2 \times 131^\circ\, 55', 704 \cdots =$ $263^\circ\, 51'{,}409{,}047{,}4$; and that for $\tan\theta = 0.001$ is $395^\circ\, 37'{,}113{,}571$. (f. 309 contains a list of m.p.a.'s for latitudes λ, where $\tan(\frac{1}{4}\pi - \frac{1}{2}\lambda) = 10^{-n}$ for $n = 1$ to 22). This scheme is not rigidly adhered to, in that an attempt is made, sometimes, to keep the added part small, for example, for 80° 0′ the m.p.a. is not 131°55′,704 &c., but 138°55′,704 &c. (f. 322).

The later calculations are somewhat simplified, for example at f. 374:

lat.	comp.	Dimid.
8$\bar{8}$. 52′	$\bar{1}$. 8′	$\bar{0}$. 34′. 00″.

℧ 98,905,216,000	
98,905,216.000	
100,000,000.	$\bar{0}$
9890,521.60	
9,892,948,48	37′
2,426,88. — 843,5	
2,877,32. — 1000,0	
26$\bar{3}$. 51′,409,0	
0. 37′,843,5.	
26$\bar{4}$. 29′,252,5.	

The value obtained is entered directly into the table (f. 192) where the first and second differences are then formed. The shortened method simply omits stage (iii) and goes straight to the linear interpolation, but for whole minutes.

Modern tables do not permit all these values to be checked, except by an independent calculation.

The mer-part for 89° 59′ is calculated in this short way (f. 398), and also with especial care at f. 399. The two values so obtained are 506° 14′,962,0 and 506° 14′,963,645,6.

The number of decimal places in the tables is usually 6, but for 0° (1′) 1° nine are given, from 86° 30′ (1′) 86° 45′ eight are given, and from 87° 0′ the number is three. We shall see that in the main body of the table the accuracy is to about 4 or 5 (usually 4) of the 6 decimal places, that is, about 1 in 10^9. The entries beyond 87° 0′ appear to be correct (with one exception) to their apparent accuracy of 3 places.

There are a number of inverse calculations. For example (f. 306):

		Lat.	Long.
♉	4,559,381,302		
	$\overline{24}$. 30′,603,6		
	2		
	$\overline{49}$. 1′,2061,2	$\overline{40}$. 58′,7938	$\overline{45}$. 0′

showing that the latitude 40° 58′,7938 has 45° for its mer-part. The use of the word "Long." corresponds to the result[77] that

$$\text{d. long.} = \text{d.m.p.} \times \tan \text{Course}$$

so that d. long. is the same as d.m.p. on the 45° rhumb. The inverse calculation is based on the sequence

$$n^\circ \rightarrow \alpha^n \rightarrow \tan^{-1}(\alpha^n) \rightarrow 90^\circ - 2\tan^{-1}(\alpha^n), \tag{24}$$

e.g.

$$\begin{aligned} 45^\circ \rightarrow \alpha^{45} &= 4{,}559{,}381{,}302 \rightarrow \tan^{-1}(0.4559\ldots) \\ &= 24^\circ\, 30',6036 \rightarrow 90^\circ - 2\times 24^\circ\, 30',6036 = 90^\circ - 49^\circ\, 1',20612 \\ &= 40^\circ\, 58',7938. \end{aligned}$$

This is similar to what HARRIOT did earlier in the *Doctrine*, except that the *α-tables* (which establish a scale) are now used.

The directly calculated mer-parts appear in tabular form in full at ff. 311–317, in pairs at the space of 1′ difference of latitude. The use of the third and fourth columns of these tables will be explained in the next section.

(vi) Calculation of the Intermediate Values

It would be pleasant to record that the intermediate values were found by the addition of secants, with allowance made for the (non-zero) second differences.[78] Unfortunately, although this does occur occasionally, more tortuous methods are usually employed when corrections are needed. This is mainly because of the increasing error in powers of the not quite exact value found for β initially. Truncation and rounding errors would have led to some difficulties, but these would not have been so great. However, one must not get the wrong impression. Although the various corrections and adjustments applied to incorrect 5th and 6th

[77] See Fig. 3b.

[78] The third differences are negligible.

decimal places produce other incorrect 5^{th} and 6^{th} decimal places, it remains true that tables accurate to 4 places (3 for very high latitudes) are produced, and that for high latitudes these remain unique, 350 years are their construction.

The following is an extract from f. 1, where the mer-parts of 0° 0′ (1′) 0° 30′ are calculated:

	2 327	7	1 ,000,032,000
[0̄].28′	1 000 033 170.—	1,163.+	28′,000,309,584
	2 412	7	1 ,000,034,369
29′	1 000 035 582.—	1,206	29′,000,343,953
	2 496	7	1 ,000,036,823
30′	1 000 038 078.—	1,248	0̄. 30′,000,380,776
		7	

The main entry in the second column is the secant of the angle in the first column, the intermediate values the differences. The main entry in the third column is half the difference next *above* in column 2. The intermediate figure, a 7, is $1/6^{th}$ the difference between the main entries. An example of the calculation is this: The 7 is added to 1,206 to obtain 1,213, which is then subtracted from 1,000,035,582 to obtain 1,000,034,369, which appears in the fourth column, where it is added to the mer-part of 0° 28′ (which has already been found in the same way) to obtain the mer-part of 0° 29′, namely 0° 29′,000,343,953.

What exactly is happening here? First note that for all angles between 0° and 1° the first difference of the secants is 84 or 85, so that (half the difference)/6 is 7, and with one exception, at 30′—31′, all the corrections of the table are 7. Neglecting these second differences is equivalent to using the linear interpolation formula for areas, $f_1 - \frac{1}{2} \nabla_1$ (measured in minutes). But this rather overestimates the area, and what is actually used is the equivalent of the (correct) formula

$$f_1 - \tfrac{1}{2} \nabla_1 - \tfrac{1}{12} \nabla_1^2, \tag{25}$$

namely LAPLACE's formula for constant second differences, usually obtained nowadays by integrating

$$f_p = f_1 - p \nabla_1 + \tfrac{1}{2} p (p-1) \nabla_1^2,$$

to obtain

$$\text{area} = \int_0^1 f_p \, dp = f_1 - \tfrac{1}{2} \nabla_1 - \tfrac{1}{12} \nabla_1^2.$$

It should be mentioned that nowhere in the nearly 675 sheets relevant to these mer-parts tables does HARRIOT explain any step he takes.[79] A consequence

[79] It has been necessary to look at all the sheets to see how they fit together. This has not been as difficult as it might have been, as many of the papers remain in more or less their original order (sometimes backwards, as if they had just left the author's desk), although with a few sheets isolated or missing. It has not yet been possible to explain every calculation that appears, although sometimes the apparently meaningless calculations have become clear, *e.g.* ff. 186 *et seq.*, where divisions for the linear interpolations of the direct calculations for 87° 17′ &c. appear. However, the present account is thought to be complete in respect of the construction of the mer-parts tables, so that the few unclear sheets may refer to false starts or fringe problems.

of this is that it is not always possible to be sure that when HARRIOT appears to be carrying out a particular process, he is consciously doing exactly what one might like to suppose he is doing. Usually there is little doubt, but not always, and so it is here with the second difference corrections. It certainly appears that an integrated form of the interpolation formula has been used, and it is true that HARRIOT was the author of the treatise on triangular numbers already mentioned,[80] which deals with interpolation. This "integrated" form is equivalent to a parabolic approximation, and the area under parabolic arcs was described by ARCHIMEDES,[81] whose works were probably the main source of inspiration of the Renaissance mathematician. Hence there is nothing unreasonable in the supposition that HARRIOT knew exactly what he was doing here, particularly as, at f. 161, where the correction is 3, which is exactly the second difference correction,[82] there is a note at the foot of the sheet

$$\boxed{\tfrac{1}{6}\,\text{diff}: = 3}$$

neatly boxed, as shown. This is one of the few places where the exact second difference correction is used.[83]

After 1° 0′, when the secants begin to be given to 6 decimal places only, there is no further need for corrections until about 59° 0′ (f. 119), although there are occasional *adjustments*. This word is used here to describe the small additions or subtractions that are necessary to bring the linear interpolation into line with the directly calculated mer-parts. For example, at the foot of f. 82,[84] the value for 41° 0′ found by repeated additions is 45° 01′,597,896, whereas by direct calculation 45° 01′,597,895 was obtained. This explains the "−1" in the margin at the foot of f. 82. This sort of thing is trivial, and these adjustments, which are listed by HARRIOT himself at f. 210, do not exceed 6 in magnitude before 75°. After that they do become quite substantial, but without ever affecting the 4th place of decimals.

The behaviour of the *corrections*, on looking at the tables, is curious. They are mostly no greater than 1 up to 76° 0′; all 3's for 80° 0′ (1′) 80° 30′, where $\nabla^2 = 31, \ldots, 36$. By 84° 0′ (f. 169) they are between 10 and 19, where ∇^2 is between 148 and 169, but they then *decrease*, which is unexpected. From 85° 16′ to 85° 30′ the correction remains *constant* at +28. HARRIOT was interested in the third differences here, as he has a table of them (f. 179v.), but these are not large enough to be important. The corrections from 85° 30′ (1′) 86° 0′ are a constant 27 as far as 85° 45′, and then increase to 43 (f. 175). From 86° 0′ (1′) 86° 31′ they increase from 44 to 63 (f. 181), and then increase further from 63·06 to 73·40 at 86° 46′, where the table is (briefly) to 8 decimal places. All this may seem

[80] *Op. cit.* (33).

[81] *Quadrature of the Parabola*, in T. L. HEATH, *The Works of Archimedes* (New York, n.d.), 233—252, or in E. J. DIJKSTERHUIS, *Archimedes* (Copenhagen, 1956), 336—345.

[82] This is the table for 80° (1′) 80° 30′, and is to six decimal places.

[83] Remember that the very slight initial uncertainty in β will throw such corrections based on correct secant tables into some confusion, particularly for the higher latitudes. However, the corrections actually used are usually rather close to these theoretical values.

[84] See *Appendix*.

rather capricious, but it is possible to reconstruct what HARRIOT was doing. Consider the following extract (f. 169):

	ψ		
$8\bar{4}.0'$	9 566 772		$16\bar{8}.\ 56'$,844,796
	26 551		9 580 037
$1'$	9 593 323	$-13\,276$	$16\bar{9}.\ 06'$,464 833
	26 700	$+10$	

The results of the last column are from direct calculation (f. 315 and also f. 324) and their difference is 9,580,037. Now,

$$\tfrac{1}{2}\{\psi(84°\ 1') + \psi(84°\ 0')\} = 9{,}580{,}047,$$

so that a correction of 10 has to be applied. To determine how this correction varies here, the directly calculated mer-parts for 84° 15′ and 84° 16′ are considered (f. 315):

$8\bar{4}.\ 15'$	$17\bar{1}.\ 23'$,450,637	9,995,669	
		10,010,147	$\psi\ 8\bar{4}.\ 16'$
$16'$	$17\bar{1}.\ 33'$,446,306	14,478	

On f. 169,

$$\tfrac{1}{2}\{\psi(84°\ 16') - \psi(84°\ 15')\} = 14{,}459,$$

so that the correction is 19. The intermediate corrections are interpolated linearly. Unfortunately, it is then found that the cumulative result of these corrections gives, for the mer-part of 84° 15′, the quantity $17\bar{1}.\ 23'$,450,610; so that an *adjustment* of +27 must be made before making the later corrections. The correction for 84° 30′ is 13, so that the intermediate corrections have to decrease, and even then it is necessary, as before, to make an *adjustment*, this time of +18.

In fact, the corrections for the higher latitudes above about 84° 0′ seem to behave in an almost random way, although we have just seen how HARRIOT determined them.[85] From 87° 0′ the entries in the tables are all obtained by direct calculation, so that there is no longer this problem.

This completes the present description of the construction of the mer-parts tables, but there are some related problems that need attention. It has already been said that the tables were ready in the earlier part of 1614.[86] This was the year of the publication of NAPIER's *Descriptio*,[87] and we shall now look briefly

[85] See (83). Some consequences of the error may be seen by comparing HARRIOT's figures with modern ones (L. J. COMRIE, *Chamber's Six-Figure Mathematical Tables* (Edinburgh, 1959), ii, 576.

α^{180}	432 139 192 210 160 87	(f. 304)
$e^{-\pi}$	0.0432 139 182 637 72	
α^{90}	2078 795 786 531	(f. 302)
$e^{-\frac{1}{2}\pi}$	0.2078 795 763 507	

[86] Or, at least, were nearing completion at that time.

[87] *Op. cit.* (33).

at HARRIOT's consideration of that work. NAPIER's tables, as is well-known, are of $10^7 \ln \operatorname{cosec} \theta$, and with the logarithm of the sine of an angle he gives that of the complement of the same angle. In a middle column, headed *differentiae,* he gives the difference of these logarithms. Thus this column is of $10^7 \ln \tan(90° - \theta)$, hence, if we replace $90° - \theta$ by $45° + \frac{1}{2}\lambda$, so that $\lambda = 90° - 2\theta$, and also take account of the scale constant $k = 10800/\pi = 3437.7 \ldots$, *we* can obtain a table of mer-parts almost directly from NAPIER's tables. To take an example, if $\theta = 30°$, then $\lambda = 30°$ also. NAPIER's *differentia* is 5403059, which when multiplied by k gives 1888',374583. HARRIOT's value (f. 312) is 1888',3754368, and the correct value is 1888',37542. Thus the value obtained from NAPIER's tables would have been about 50 times as much out as was HARRIOT's nearly correct figure. But NAPIER's tables err in the 7^{th} place,[88] and when reissued in 1616 were cut down to 6 figure tables.

It is worth noting that BM Add MS 6786 contains calculations related both to Napierian and Briggsian logarithms. In particular, at f. 84, there is:

Logarithmus noster 1'	81,425,715
Neperi:	81,425,680
	35 differentia
noster 2'	74,494,239
Neperi:	74,494,211
	28 differentia

Now, HARRIOT's first value is exactly correct, and his second is 5 units too small,[89] so that his values are better than NAPIER's. Too much should not be made of this, as these are only the first two entries of NAPIER's tables.[90]

(vii) *The Tables of d. longs*

Leconfield 240, ff. 263—269, contains a complete set of tables of d. longs tabulated for latitudes 0° (1°) 89° and 89° 59' on the seven rhumbs. The calculations for these exist in their entirety.[91] A few questions related to rhumbs, arising from *ibid.*, f. 236, may be considered here. The periphery of a quadrant of a circle is given there (correctly) as 15,707,963,267, ... This is then squared, and

[88] JAMES HENDERSON, *Bibliotheca Tabularum Mathematicarum* (Cambridge, 1926), 24—25.

[89] NAPIER's first value ends with 681, and not 680, in the copy seen.

[90] From the dating of papers bound in the same volume, whose order has most likely been disturbed since the 1620's, it is reasonable to suppose that HARRIOT's results just quoted were of shortly after the appearance of NAPIER's book, say 1614 or 1615, rather than 1619 when the *Constructio* appeared. f. 87 shows HARRIOT looking at both NAPIER's logarithms and his own mer-parts. f. 200 has a table of successive square roots, and other folios, *e.g.* f. 197 and ff. 213—214 appear to be related, and also connected with Leconfield 240, f. 238. Work on this is at present incomplete.

[91] BM Add 6786 1st rhumb: ff. 29v., 149, 149v., 160, 160v.; 2nd: ff. 150—152v.; 3rd: ff. 30—32v.; 5th: ff. 33, 33v., 34, 161, 161v.; 6th: ff. 2, 3, 29, 154, 154v.; 7th: ff. 6—7v., 9, 9v., 22v., 23, 23v. Those for the 4th rhumb are identical to the mer-parts themselves, and there is a table of these at f. 164 at 1° intervals. Whether any mariner used the d. longs. tables, I do not know, but DEE had constructed some, and both WRIGHT and GUNTER published theirs.

we read that "$\frac{1}{4}$ eius, est area rumbi $4\overline{5}$ in sphaera", that is, the area swept out between the pole and the 45° rhumb from the equator is $\pi^2/16$ for a sphere of unit radius (Fig. 12). The next line tells us "longitudo rumbi $4\overline{5}$ in sphaera. 22,214,414,690,aaa, ... &c." Now the second result, for the length of the rhumb, is correct, and follows from the result *d. lat.* = *distance* × *cos Course*. The d. lat. is $\frac{1}{2}\pi$, and so the distance is $\frac{1}{2}\pi \sec \frac{1}{4}\pi$, which is correctly calculated. But the surface area result is quite wrong, and it is not clear from where it might have come. It can be definitely stated that the correct determination of this result would have been an astounding achievement for this period.[92] As the d. long. in radians is $\tan\alpha \ln\tan(\frac{1}{4}\pi + \frac{1}{2}\lambda)$ (Fig. 12), it follows that the area enclosed, A, is given by

$$\tan\alpha \int_0^{\pi/2} \cos\lambda \ln\tan\left(\tfrac{1}{4}\pi + \tfrac{1}{2}\lambda\right) d\lambda,$$

so that, after some reduction,

$$A \cot\alpha = \lim_{\lambda\to 0} \left(\ln\sin\lambda - \cos\lambda \ln\tan\tfrac{1}{2}\lambda\right).$$

Hence

$$A = \ln 2 \tan\alpha. \tag{26}$$

There can be no doubt that HARRIOT was on the wrong track here, and with no real hope of finding the correct path. At the foot of f. 236 he correctly finds the

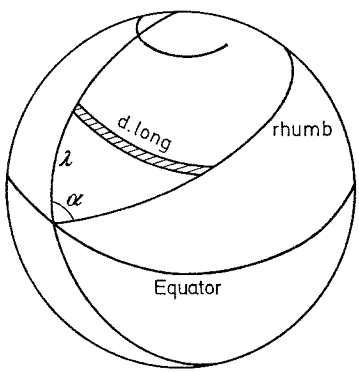


Fig. 12. Area enclosed by a rhumb line

length in nautical miles of the complete rhumb of 45°. It is about 7,636′ or 127° 16′, and this is interesting because it is a finite length for a curve which has an infinite number of turns, as well as being a twisted curve. He then attempts the same sort of thing for the incorrect area:

′	″	‴	⁗
1,256,637,061	21,600	61,685,027	1060′ 2875/10000

[92] Of course, his mer-parts construction was itself a considerable achievement, and it is dangerous to make categorical statements about what a man did not do.

that is, $4\pi r^2 : 360^\circ :: \pi^2 r^2/16 : 17^\circ\, 40' +$. This $17^\circ\, 40' +$ is described as "Area rumbi ... in sphaera". But he is beyond his (very considerable) abilities, and it is doubtful if there was anyone who could have solved this particular problem before the end of the century.

A number of isolated pieces of work found among the Leconfield 240 papers are now briefly considered.

(i) Many first stage divisions are at BM Add 6786, *e.g.* at f. 165 the work for Leconfield 240, f. 392 is given. In the entry

$$\begin{array}{lll} 8\bar{9}.\ 2' & 84{,}359{,}582{,}000 & \\ & 85{,}463{,}600 & (=\alpha^9) \\ & \ 9{,}870{,}819{,}0 & \end{array}$$

are given the 2nd, 3rd, and 4th lines of the f. 392 calculations.[93]

(ii) From f. 239, where the arcs and chords for $1^\circ, 1', 1'', \ldots, 1^{\mathrm{v}}$ are tabulated, it is seen that arc $1^{\mathrm{v}} =$ chd 1^{v} to 18 significant figures, that is, 28 decimal places. On f. 209, $\sin \frac{1}{2}^{\mathrm{v}} = \frac{1}{2}\,\mathrm{chd}\, 1^{\mathrm{v}} = 1{,}122{,}253{,}891{,}457, \ldots$ is used to find $\cos \frac{1}{2}^{\mathrm{v}}$ by the relation $\cos \frac{1}{2}^{\mathrm{v}} = \sqrt{(1 - \sin^2 \frac{1}{2}^{\mathrm{v}})}$.[94] Then $ae + eb$, kb, ak, af, bf, and ag are written down (Fig. 13), and finally the area of the triangle abf.[95]

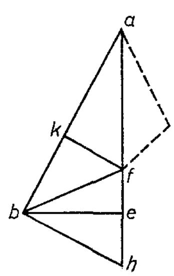


Fig. 13. The elementary triangles

(iii) ff. 249—251 appear to give a construction for the *Problem* Given a circle whose diameter ac is extended to b, find a line ag cutting the circle in h, and such that $ab = ag$, where $hc = hg$. The solution seems to be: Let yd be a diameter perpendicular to ac. Draw a semicircle on ad as diameter, and choose f on this semicircle such that $df = \frac{1}{2} ab$. Join af to meet circles in h and g. Then $ch = hg$. The solution is not stated explicitly, and the use of the many other construction lines is not clear.[96]

[93] This may seem rather cryptic, but will, it is hoped, be of use to anyone who cares to inspect the MSS for himself. The same applies to some of the later items.

[94] The approximation $(1 - e)^{\frac{1}{2}} = 1 - \frac{1}{2} e$, which would be very convenient here, is not used.

[95] kb comes from $(ae + eb)/ab = eb/bk$, which is shown at f. 252.

[96] There are copies at Harley 6001, f. 19.

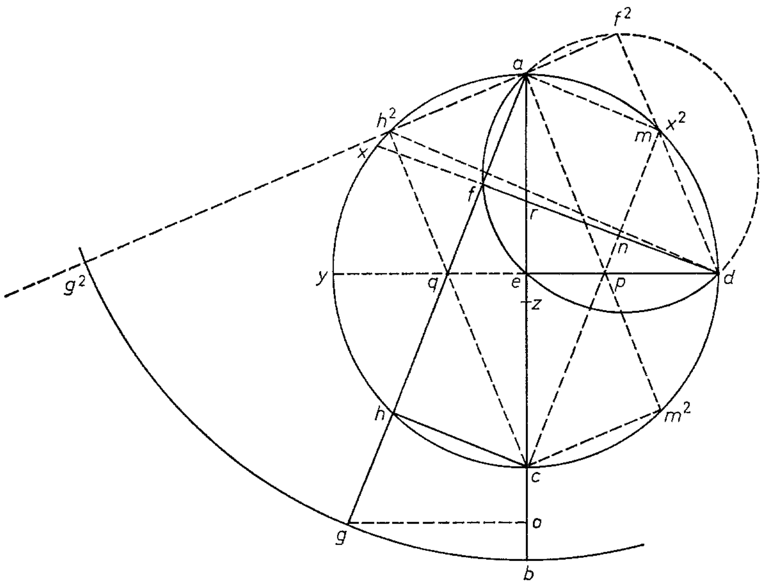


Fig. 14. Leconfield 240, f. 250

(iv) f. 254 gives a calculation of $af \cdot ae$, where $af = b - c + d$, and $ae = f - g + h$. f. 254v. calculates $af^2 = (b - c + d)^2 = ag$. This agrees with f. 209. $kf \cdot ab$ is given, and hence the area of the triangle abf ($= \frac{1}{2} kf \cdot ab$), again agreeing with f. 209.

(v) f. 320 gives the secants of 88° 0′ (1′) 88° 6′ with the note *lunthv*[97] and also some other secants with *Rheticij* and *Pitiscij*. These refer to the tables mentioned earlier.[98] The two sets do not entirely agree. 1st, 2nd, and 3rd differences are given, and the 3rd differences for *lunthv* are irregular, unlike the others.

(vi) ff. 403–405, headed *De rumbis*, have spherical triangle calculations.[99]

(vii) At f. 419, the repeated exponentiation of the length of *am* leads to radii vectores for angles up to 720°, but these do not agree very well with the values of the *fundamental tables*.[100]

(viii) At ff. 439, 440 and 431 the assumption being tested is that an increase in latitude of 1′ from the equator, on the fourth rhumb, corresponds to a 1′ increase in longitude. This is, of course, not a correct assumption, but is only made to see what it leads to. On its basis a comparison is made of the length of the straight line approximating the spiral to the decrease in radius vector. This should be very near the exact ratio of $\sqrt{2}:1$; a value of 14,142,135,474,15 is obtained, which errs in the 9th figure. A slightly different assumption (at the top of f. 431) obtains another approximation, also differing from the correct value in the 9th figure.

[97] What does this mean?

[98] *Op. cit.* (76).

[99] Is HARRIOT's work on spherical trigonometry related to that of his friend NATHANIEL TORPORLEY, *Diclides Coelometricae* (London, 1602)?

[100] *E.g.*, 186,487,007,688,*91* (f. 238) and 18,674,428 (f. 306), both for 360°.

(ix) f. 419 explains the number ($am =$) 99,970,909,029,040, which is prominent at f. 238 and in BM Add MS 6786. It is the second radius vector for the approximation of $\frac{1}{2}'$. There is a reference to sheet *c*.1, but this sheet appears to be missing.[101] The value of *bf* comes from f. 209, for $\frac{1}{2}^{v}$. Finally, among the calculations at f. 429, for the chord of $1'''$, appears the value of π, written as 31415926535897932626433 with the comment "Ludolph van Cille*n* hath a cipher to many fol 16.b".[102]

In concluding this account of HARRIOT'S calculation of the meridional parts or logarithmic tangents, it may not be out of place to express admiration for the results that he achieved. He had calculated a highly specialised form of logarithm table contemporaneously with, and without any reasonable doubt, independently of, the work of JOHN NAPIER, and in doing so had not only provided a complete solution of one of the most pressing of navigational problems,[103] whose importance has not diminished in the modern world, but also shown himself to be ahead of the mathematicians of even the later 17th century in the depth of his method, which, despite the complexity of its details, cannot fail to impress by the natural sequence of its development. That one of these details was not quite perfect, and thus led to slightly more complication than was necessary, is a small blemish, and one which HARRIOT himself could not have failed to notice. Further, the method, once devised, with its dependence on equally outstanding original work on stereography, spirals, and interpolation, did not remain just a theoretical possibility, but the necessary work for its practical realisation was carried out, despite the handicap of ill-health. It is not too much, then, to claim a leading place in early 17th mathematics for HARRIOT'S work on the direct calculation of the meridional parts.

Acknowledgements. I particularly wish to record my thanks to Lord EGREMONT, who has allowed me to study the HARRIOT manuscripts in his possession at Petworth House, Sussex,[104] without which kindness this article could not have been written. Thanks are also due to the following for providing encouragement, facilities or advice: Dr. ANGUS ARMITAGE, Professor T. A. A. BROADBENT, Dr. J. W. SHIRLEY, Mr. F. W. STEER, Dr. R. C. H. TANNER, Commander DAVID WATERS, R. N., and Dr. D. T. WHITESIDE.

101 As will appear in the *Calendar* of these papers in the *Appendix*, some of the sheets have foliation of this type.

102 VAN CEULEN'S foliated work *van den Circkel* (Delf, 1596) does not have this. His later works appear to be paginated.

103 Other problems dealt with by HARRIOT are described in my *Dissertation, op. cit.* (26), which also includes a biographical sketch.

104 Leconfield MSS 240 and 241.

Appendix

There follow transcriptions of passages that are too lengthy to be placed in the main text. Most of the passages have been referred to earlier. A *Calendar* of some of the papers considered is given. The transcriptions placed here are only a small selection of those available, but it is hoped that they are sufficient to illustrate the main text adequately. I have tried to follow uniform editorial principles[104a] and also to put down what I have seen (or thought I have seen), although this may, for example, lead to some apparently wrong Latin words.

Calendar of the Papers of the Meridional Parts Calculations

Leconfield 240 was bound about 1955 by HUGH WYNDHAM, 4th Baron Leconfield, after the removal of the sheets which form it from the early 17th century leather box in which it had been kept since after HARRIOT's death, and which was by then considerably worm-eaten.[105] Before that, the papers had been pinned together in groups, and these groups have been recorded by the then Archivist, Miss G. M. A. BECK, on papers now kept with the manuscript.[106] The folios are numbered 1—453, but ff. 233 and 234 do not exist, and there is an unfoliated sheet, which I shall call f. 224*, from its position. The papers are bound in five volumes, and the following list gives both the volumes and the groups mentioned above.

Vol. I.	ff. 1—170.	Tables of mer-parts for 0° (1′) 85° 0′.[107]
Vol. II.	ff. 171—176.	Corrections &c. for 85° (1′) 86°.
	ff. 177—183.	The same for 86° (1′) 87°.
	ff. 184—190.	Final calculations for 87° (1′) 88°.
	ff. 191—198.	The same for 88° (1′) 89°.
	ff. 199—208.	The same for 89° (1′) 89° 59′.
	f. 209 is *a*.4.	
	f. 210	The table of *adjustments*.
	ff. 211—231.	Papers *de Helicis*.
	ff. 232—239.	Various (ff. 233, 234 removed).
	ff. 240—247.	Algebra for *de Helicis*.
	f. 248.	The same.
	ff. 249—253.	The same.
Vol. III.	ff. 254—261.	Calculations on the elementary triangle.
	f. 262.	Tables of mer-parts at 1° intervals.
	ff. 263—271.	270, 271 are as f. 262. The remainder is d. longs. on the 7 rhumbs.

[104a] Such as are used in G. E. DAWSON & L. KENNEDY-SKIPTON, *Elizabethan Handwriting* (New York, 1966).

[105] The papers themselves are, however, in excellent condition. The other HARRIOT papers at Petworth, Leconfield 241, had been kept separately, and were also bound at the same time.

[106] Dr. J. W. SHIRLEY, who saw the papers in their original form, in 1947, tells me that the pins appeared to be the original ones.

[107] These were also pinned in groups, as the pin-holes show, but this has not been recorded, and, in this case, is not important.

	ff. 272—281.	272—276 are the γ-tables, and the remainder is preliminary work for this.
	ff. 282—291.	282 is as f. 262. The rest are blank tables.
	ff. 292—300.	297—300 are the β-tables. The remainder are extracts from mer-parts and α-tables.
	ff. 301—310.	301—305 are the α-tables. The other folios include direct and inverse calculations.
	ff. 311—321.	311—317 are extracts from direct calculations used in finding the *adjustments*. The rest include some divisions for the direct calculations and extracts from tables of secants.

Vol. IV. This has all the direct calculations, but does not include the routine divisions.

	ff. 322—326.	For 80° to 89°.
	ff. 327—331.	For 70° to 80° (omitting 72° and 72° 1′).
	ff. 332—336.	For 50° to 59°.
	ff. 337—342.	For 60° to 69°, and 72°.
	ff. 343—347.	For 30° to 39°.
	ff. 348—352.	For 40° to 49°.
	ff. 353—357.	For 20° to 29°.
	ff. 358—362.	For 10° to 19°.
	ff. 363—367.	For 0° to 9°.
	ff. 368—374.	From 88° 2′ to 88° 59′.
	ff. 375—382.	From 87° 1′ to 87° 59′.
	ff. 383—386.	From 83° 15′ to 86° 46′.
	ff. 387—391.	From 81° 30′ to 88° 59′.
	f. 399.	For 89° 57′, 58′, and 59′.
	ff. 400—402.	Cancelled calculations.
Vol. V.	ff. 403—405.	*de Rumbis* 1, 2, 3.
	f. 406.	Rough calculations.
	ff. 407—417.	*Ad calculum sinuum.*[108]
	f. 418.	*de Helicis.*
	ff. 419—423.	Numerical calculations on the spiral. f. 423 is *de Helicis* $\overline{45}$.
	ff. 424—432.	The same, including β (f. 432).
	ff. 433—438.	Algebra and arithmetic on the spiral.
	ff. 439—440.	Diagrams and arithmetic on the spiral.
	ff. 441—445.	Calculations for m.p.a. = 0.1.
	f. 446.	The same.
	ff. 447—453.	Sheets *e.*1, and *d.*2—*d.*7. (*d.*1 is at f. 238), powers of 99,970,909,029,040 for 1′ to 720°.[109]

[108] These are not related to the mer-parts, but may be connected with papers in the British Museum, see (27), and Sion College (arc. L.40.2/L40, arc. L.40.2/E10, and arc. L.40.2/E6; these are the TORPORLEY papers, and the first volume may be relevant).

[109] *Cf.* BM Add MS 6786, ff. 181—182, 213—214.

BM Add MS 6786, ff. 1—217 are relevant. Many of the folios have been referred to. With a few exceptions the work is for calculations of the d. long. tables, multiplications for the *fundamental tables*, and divisions for the direct calculations.[110]

Table 2. *Cases of the nautical triangle. (Leconfield 241, vi.b, 6)*

— 6 —

9. The propositions praemised beinge sufficient to express the nature of the nauticall triangle; The methode for practice I order thus:

The rectangle beinge alwayes geven I say also muste be knowne the:

1. Meridian segment & Angle of direction	And then the thing sought is	1. The line of distance 2. The difference of longitude
2. Meridian segment & distance	The thing sought	2. Difference of longitude 1. Angle of direction
3. Meridian segment & Diff: of longitude	The thinge sought	1. Angle of direction 2. Distance
4. Angle of direction & distance	The thing sought	1. Merid: segment 2. Difference of longitude
5. Angle of direction & diff: of longitude	The thing sought	[1.] Merid: segment [2.] Distance
6. Distance & Diff: of longitude	The thing sought	[2.] Merid: segment [1.] Angle of direction

Table 3. *Solution of the nautical triangle (ibid, 8, 10)*

— 8 —

The proportionall termes of the first order.

	I	II	III	IIII
1.	Whole sine 10000	Degrees of ye segm: of ye meridian	Secant of ye angle of direction	Degrees of distance
	I	II	III	IIII
2.	Whole sine 10,000	Aequall degrees awnswerable to the vnequall of the Meridian	Tangent of the angle of direction	Degrees of longitude or arke of a parallel

110 There are also some sheets *de logarithmis*, ff. 81—84, 87—91; for ff. 181—187, 197—217, which appear to do with logarithms of Briggs' type, more consideration will be necessary. Finally, BM Add 6788, f. 466, gives the latitudes for the first seven turns about the globe on the 7th rhumb as

$$58^\circ\,1',170;\ 80^\circ\,36';\ 87^\circ\,18';\ 89^\circ\,13';\ 89^\circ\,46';\ 89^\circ\,56',\ 89^\circ\,58'.$$

The Second Order

	I	II	III	IIII
1.	Degrees of the segment of ye meridian	Whole sine 10000	Degrees of distance	secant of the angle of direction
2.*	Degrees of the segment of ye Meridian	A*e*q: Degrees awnswerable to ye vnequall of the Merid.	Degrees of distance	Degrees awnswerable to the supposed vnequall of distance

for the second parte of this order the spediest way to find the thing sought is by the second sorte of termes of the first order: thus:

	I	II	III	IIII
2.	Whole sine 10,000	Aequall degrees answerable to ye vna*e*quall of the Meridian	Tangent of the angle of direction	Degrees of long: or arcke of a parallel

The Third Order.

1.	Aequall degrees awnswerable to ye vnequall of ye meridian	Whole sine 10000	Degrees of diffe: of longitude	Tangent of ye angle of direction

2. The second parte hath the termes of the first parte of the first order.

— 10 —

The fourth order.

	I	II	III	IIII
1.	Whole sine 10,000	Degrees of distance	Sine of ye complement of ye angle of direction	Segment or degrees of the meridian

2. The termes of the 2 of the first order

The fifth order.

	I	II	III	IIII
1.	Whole sine 10,000	Degrees of longitude	Tangent of the complement of ye angle of direction	a*e*quall degrees answerable to the vnequall of the segment of the Meridian

2. The first of the 1 order

* This analogy is cancelled in the manuscript and replaced by the one below it.

— 19 —

Arcus	1	9,826,974
tangentium.	2	19,653,948
	3	29,480,922
Continue	4	39,307,896
proportionales	5	49,134,870
Tangentes.	6	58,961,844
	7	68,788,818
	8	78,615,792
	9	88,442,766

10,000,000 9,826,974	44.30′.0″	8$\bar{9}$.0′.0″	$\bar{1}$.0′.0″	60,*0031,225*
9,656,942	44.0. $\frac{53}{5623}$	88.0′. $\frac{106}{5623}$.	$\bar{1}$.59′. $\frac{5517}{5623}$.	120,0062 450 98175
			$\bar{1}$.59′.0.	119 02449. 42 +.
1. 9,997,093	4$\bar{4}$.59′.30″	89.59′.0″	0.1′.0″	1,000,000,0

2. 9,994,187
3. ———
4. 9,988,377

———
———
———

8. 9,976,767
16. 9,953,588
32. 9,907,391

				2688.
64. 9,815,640	4$\bar{4}$.28′. $\frac{95}{5713}$.	8$\bar{8}$.56′. $\frac{190}{5713}$	$\bar{1}$.3′. $\frac{5523}{5713}$	64,000,000,0.
			$\bar{1}$.4′.	64,033,263,2.
				03 7

1. 9,997,090,0
2. 9,994,180,8
4. 9,988,364,9
8. 9,976,743,3
16. 9,953,340,6
32. 9,907,297,0
64. 9,815,453,3

— 21 —

1	9,826,974
2	19,653,948

(Leconfield 241, vi.b)* — 13 —

Analogia ad examinanda*m* numeratione*m* Canonis Nautici nostri.

I	II					
Radius seu Tangens $\overline{45}$ 10,000,000 / 7	Tangens	$44\frac{1}{2}$	9,826,974	$\overline{1}$	compl.	89
		$\overline{40}$	8,390,996	$\overline{10}$		80
		$37\frac{1}{2}$	7,673,270	$\overline{15}$		75
		$\overline{35}$	7,002,075	$\overline{20}$		70
		$32\frac{1}{2}$	6,370,702	$\overline{25}$		65
		$\overline{30}$	5,773,502	$\overline{30}$		60
		$\overline{25}$	4,663,081	$\overline{35}$		55
				$\overline{40}$		50
			semisses			
		$42\frac{1}{2}$	9,163,312	$\overline{5}$		$8\overline{5}$

1	8,390,996
2	
3	
4	
5	
6	
7	
8	
9	

1	8,390,996 40
2	16,781,992 80
3	25,172,989 20
4	33,563,985 60
5	41,954,982 00
6	50,345,978 40
7	58,736,974 80
8	67,127,971 20
9	75,518,967 60

Consectariu*m*.

Si fit multiplex sectio rumbi per lineas a centro secundu*m* angulos a*e*quales linea*e* a centro sunt continuè proportionales, geometricè.

Hinc fit numeratio facilior: vt in superiori exemplo.
Quadra 8,390,996. Quadratu*m* est. 7040881*3872016*.
divide per 10,000,000, vel quod ide*m* est reijce 7 primeas figuras et numerus est 7040881. ideam vt residuu*m* in Quarto.
Quo cu*m* numero opera vt ibi.

---

2. Arcus paralleli, ducti a terminis linearu*m* illaru*m* proportionalium, (mensurantes angulos a*e*quales,) sunt etiam continue proportionales in eadem ratione.

3. Item, sectiones rumbi subtendentes illis angulis sunt in eadem ratione.

* As the pages 13 and 14 of this manuscript should be read together, it has been necessary to print the transcription of the succeeding pages 19 and 21 out of order, and the reader is asked to turn to page 395 for it.

— 14 —

III			IIII
Dua*rum* Tangentiu*m*	44½.		Quartus proportionalis
45̄ et ...	40̄.	1,609,004	40̄. 1,350,115.
100000	37½.		subtrahe a Tangente. II.
	35.		Residuu*m*. 7,040,881.
	32½.		Residui, in canoni tangentiu*m*
	30.		qua*e*re arcum.
	27½.		35̄.9′.
	25.		Arcum dupla. 70̄.18′.
			90.00.

Duplati nota complementu*m*. Qui numerus, est latudo, in qua dupla longitudo invenitur ad primam, termini scilicet rumbi 45̄. etct.

19̄.42′.

10,000,000 89.60

8,390,996	40.0′	80̄. 0.	10̄.	603′,*088,252*	
7,040,881	35.9′	70̄.18′	19̄.42′.	1206′,*176,504*	
5,908,000	30.34′½	61̄. 9′	28̄.51′.	1809′,*264,756*	
4,957,400	26.22′$\frac{1}{12}$	52̄.44′.10″	37̄.15′.50″.	2412′,*353,008*	
4,159,752	22.34′.50″	45̄. 9′.40″	44̄.50′.20″.	3015′,*441,260.*	50̄.15′
8,390,996,40 7,040,882,05 5,908,091,59 4,957,477,52					
7,132,931	35̄½	71̄.0′	19̄.0′	1161,*515791*	
			36.3′	2323,*031582*	
5,087,869.	26.57′$\frac{3602}{3661}$		36.4′$\frac{128''}{3661}$		

Extracts from Leconfield MS 240

Tables of meridional parts

(f. 82)

4$\overline{0}$. 30′	1 315 087 —		4$\overline{4}$ 21 997 019
	327		1 315 251
31′	1 315 414 —	— 163 +	23 312 270
	327		1 315 577
32′	1 315 741 —	— 164 —	24 627 847
	327		1 315 905
33′	1 316 068	— 163 +	25 943 752
	328		1 316 232
34′	1 316 396	— 164	27 259 984
	328		1 316 560
35′	1 316 724	— 164	28 576 544
	328		1 316 888
36′	1 317 052	— 164	29 893 432
	329		1 317 216
37′	1 317 381 —	— 165 —	31 210 648
	329		1 317 546
38′	1 317 710 —	— 164 +	32 528 194
	329		1 317 874
39′	1 318 039 —	— 165 —	33 846 068
	329		1 318 204
40′	1 318 368 —	— 164 +	35 164 272

[&c.]

(f. 169)

8$\overline{4}$. 0′	9 566 772		16$\overline{8}$.56′884 796
	26 551		9 580 037
1′	9 593 323	— 13 276	169.06 464 833
	26 700	+ 10	9 606 663
2′	9 620 023	— 13 350	16 071 496
	26 849	10	9 633 436
3′	9 646 872	— 13 425	25 704 932
	27 001	11	9 660 361
4′	9 673 873	— 13 501	35 365 293
	27 153	11	9 687 437
5′	9 701 026	— 13 577	45 052 730
	27 307	12	9 714 666

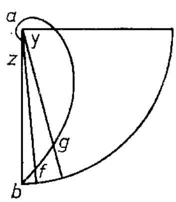
[&c.]

(f. 199 gives the mer-parts for 89° (1′) 89° 30′. In the following extracts, the differences (which go as far as the fourth) are omitted.)

$8\bar{9}.\ 0'$	$27\bar{1}.\ 39'\ 556$	$(89.)\ 5'$	$27\bar{6}.\ 38\ 639$
$1'$	$27\bar{2}.\ 37'\ 338$	$6'$	$27\bar{7}.\ 41\ 775$
$2'$	$27\bar{3}.\ 36\ 107$	$7'$	$27\bar{8}.\ 46\ 037$
$3'$	$27\bar{4}.\ 35\ 898$	$8'$	$27\bar{9}.\ 51\ 522$
$4'$	$27\bar{5}.\ 36\ 747$	$9'$	$28\bar{0}.\ 58\ 279$

(f. 200, the mer-parts for 89° 30′ to 89° 59′, includes the following figures. Again the differences (which go up to the fifth) are omitted.)

$(8\bar{9}.)\ 50'$	$37\bar{4}.\ 19\ 256$	$(8\bar{9}.)\ 55'$	$41\bar{4}.\ 02\ 123$
$51'$	$38\bar{0}.\ 21\ 459$	$56'$	$42\bar{6}.\ 49\ 233$
$52'$	$38\bar{7}.\ 06\ 368$	$57'$	$44\bar{3}.\ 18\ 212$
$53'$	$39\bar{4}.\ 45\ 415$	$58'$	$46\bar{6}.\ 32\ 099\ 0$
$54'$	$40\bar{3}.\ 35\ 347$	$59'$	$50\bar{6}.\ 14\ 963\ 6$



(f. 221) 6. De longitudine Helicis $4\bar{5}$. Vnius et prim*ae* revolutionis.

az. 18,708,345 vt in 5ta charta ca*e*tera esto vt in 1a.

ab	*bf*	*az*	*zy*
10,000,000,000.	4,112,584.	18,708,345.	$7{,}693.\ \frac{96}{100}$

sit prima proportionalis *bf*, et notetur α. *fg*, secunda, β. *zy*, minima, χ. *bfgzy*, ω.

$$\frac{\alpha\alpha-\beta\chi}{\alpha-\beta}\quad \omega. \quad \text{vel } bfgz + zy.$$

$\alpha\alpha$.	16,913,347,157,056.	vt in prima charta.
$\beta\chi$.	− 31,628,907,884	
$\alpha\alpha-\beta\chi$.	16,881,718,249,172.	

α min*us* β · 1196, vt in 1a charta.

ω.	14,115,*aaa,aaa*	
zy.	7,693	
bfgz.	14,115,*aaa,aaa*.	composita ex rectis.

ab	Tota Helix	*az*	Residua *a, z, ω, a*.
10,000,000,000	14,142,135,623	18,708,345	26,457,595

	14,142,135,623
bfgz.	26,457,595
	14,115,678,028.

Helix curva et eius vera longitudo si sectio sit in *z*, vt videtur. In tot signis,[111] quibus consentit cu*m* composita ex rectis, est secans $4\bar{4}.\ 53'.\ 30''$ scilicet per 5 primas figuras.

[111] Thus the copyist in Harley 6001, f. 21v. The original looks like *figuris*.

Si lineas insistentes in Helicen a*e*qualiter excedent vel deficiant intersegmenta. helicis sunt a*e*qualia.

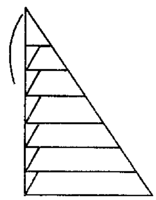
Corollariu*m*.

Hinc sequitur intersegmenta helicis, qua*e* sunt inter parallelos a*e*quidistantes sunt a*e*qualia.

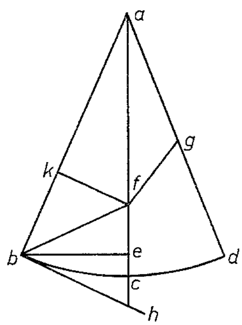
Theorema.

Si sint ⟨magnitudia⟩ quantitates quotcunq*ue* proportionales et illa*e* qua*e* su*nt* alterna*e* a*e*qualiter excedant vel deficiant altera*e* alterna*e* sunt a*e*qualiter excedentes vel deficientes vel si non a*e*qualiter excedant vel deficiant differentiae (pra*e*dictaru*m*) sunt proportionales.

12. 18
10. 15
8. 12
6. 9



(f. 224)



De longitudine Helices 45gr.

Arcus *bc*. *bd*		*abf*.	$4\bar{5}$. 0′
vel Angulus. *bac*. 1′		*afg*.	
ab.	10,000,000,000		
bc	2,908,882		
ae.	9,999,999,577	*abe*.	$8\bar{9}$. 59′
			$4\bar{5}$. 0.
fe.	9,994,183,927.	*fbe*.	$4\bar{4}$. 59′
be.	10,000,000,000.	*bfe*.	$4\bar{5}$. 1′
bf.	14,138,025,637.		

be.	*be*.	*fe*.	*fe*.
10,000,000,000	2,908,882	9,994,183,927	2,907,190
			2,907,190,*2185*
	2,908,882,*0456*	*bf*	*bf*
		14,138,023,637	4,112,584
			4,112,584,*3118*

	ae.	9,999,999,577	
	fe.	2,907,190	
Ergo.	*Af*.	9,997,092,387	
	ag.	9,994,185,619	4,213,357,769
		$\bar{1}$	0 000 000 000

ab.	*bf*.	*af*.	*fg*.
10,000,000,000	4,112,584	9,997,092,387	4,111,388,*5313*

Sint continue proportionales.

bf. prima. *fg*. secunda. Et ita in infinitu*m*. Quaeriter sum*m*a.

Notatio Logistica.

α pro prima. β pro secunda. &c. ω sum*m*a.

$$\frac{\alpha\alpha \langle -\beta \rangle}{\alpha - \beta} = \omega.$$

α.	4,112,584		4,112,584,*3118*
β.	4,111,388	vel melius	4,111,388,*5313*
$\alpha - \beta$	1,196		1,195,7805
$\alpha\alpha$.	16,913,347,157,056		
ω.	14,141,5*aa,aaa*.	Transinuosa	$4\bar{4}$. 59′. 51″.

ω. est quantitas Helices composita ex rectis.

Accurata quantitas Helices curva
14,142,135,623.

(f. 229) 4. De Area inclusa per Helicen $2\bar{2}$. 30′.

be	2,908,882
be/2	1,454,441
fe	7,016,888

$\left.\begin{array}{l} be/2 \;\; 1{,}454{,}441 \\ fe \;\; 7{,}016{,}888 \end{array}\right\}\square$ — Area $\triangle^i$ *bfe*. — 10,205,649,599,608

ae	9,999,999,577.
be/2	1,454,441.

$\left.\begin{array}{l} ae \;\; 9{,}999{,}999{,}577. \\ be/2 \;\; 1{,}454{,}441. \end{array}\right\}\square$ 14,544,409,384,*771*,*457*. area $\triangle^i$ *abe* vt antea in 2a charta

Ergo area $\triangle^i$ *abf*. — 14,534,203,735,171,849

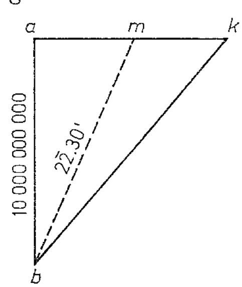
ab. $\square$.	*af*. $\square$.	*abf*. $\triangle$.	*afg*. $\triangle$.
100,000,000,000,000,000,000	99,859,703,022,653,670,721.	14,534,&c.	14,513,812,686,650,045.

Sint continue proportionales in infinitum.
prima $\triangle$. *abf*, et notetur α. Secunda $\triangle$ *afg*, et notetur β.
Quaeritur summa. ω.

$$\frac{\alpha\alpha}{\alpha - \beta} = \omega.$$

α.	14,534,203,735,171,849.
β.	14,513,812,686,650,045
$\alpha - \beta$.	20,391,048,521,804
$\alpha\alpha$	211,243,078,215,483,326,980,341,562,078,801.
ω	10,359,544,aaa,aaa,aaa,aaa. Triangulorum rectilineorum summa.

Accurata quantitas inclusa per Helicen.
10,355,339,060,000,000,000.
videlicet semissis $\triangle^i$ *abk*.
Tangens enim *ak*. 4142135624
ak/4 ————. 10,355,339,06.
$\left.\begin{array}{l} ak/4 \\ ba \end{array}\right\}\square$



De Sum*m*a proportionaliu*m* *ab*. *af*. *ag*. &ct.
sit prima, α. Secunda, β. Sum*m*a, ω.

$$\frac{\alpha\alpha}{\alpha-\beta} = \omega.$$

α.	10,000,000,000
β.	9,992,982,689
$\alpha-\beta$.	7,017,311
$\alpha\alpha$	10,000,000,000,000,000,000,0.
ω.	14,250,472,aaa,aaa.

Tang. $8\bar{9}$. 57'. 40''
secant: sed qua*e*re.

(f. 230) *De Rumbis.*

Seeing that the rumb of 45 degrees is equal to the tangent of $4\bar{5}$. or the side of the square inscribed. I meane in a circle Then if that circle be understoode to be a greater circle, & one pole to be the vertex of a cone. I meane of such a cone as ⟨wilbe⟩ I will describe that is; a line infinite being supposed to have one end fixed in the sayd pole & then to move fro*m* the ⟨end of⟩ beginning of the rumbe in the circle till he passe all the rumbe to the center. According to the doctrine of our planisphere it also describeth a rumbe on the contrary superficies of the sphere. The superficies made by this ⟨the⟩ line so moued I call a Rumbicall Superficies. The rumbe it self, is the circumference of the base of the cone. The Superficies comprehended by a right line from the beginning of the rumbe to the center of the circle: or by the arke of a greate circle fro*m* the beginning of the rumbe to the other pole: & the rumbe: is the base of the cone. And the Rumbicall cone is that which is comprehended betwene the Rumbicall Superficies & the sayd base. Now there are two cones to be vnderstoode having one vertex. the one hath his base in the sayd circle & the other in the superficies of the sphaere.

I demande what proportion hath ⟨there⟩ the circumference of there bases or two rumbes?

What there conicall superficies?

What there solides?

What the superficies of there bases?

What the semidiameters of there bases? that is, the proportion betwixt the semidiameter of the circle & a quarter of the periphary?

The scope & purpose is to finde some other elements for the Doctrine of the rumbes, & to consider whether by that way may be found any thing concerning the quadrature of the circle.

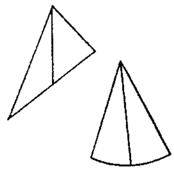
Let this cone also be considered: that hath his vertex in the center of the sphaere & the base in the superficies of the spha*e*re as above.

(f. 231) sint continue proportionales insistentes in Helicen et distantes per amplitudine*m* vnius minuti.

ab notatur, β. prima. *af* notatur α secunda &ct. ω sum*m*a.

$$\frac{\beta\beta}{\beta-\alpha} = 34,377,\&ct.$$

If a triangles base be devided a*e*qually by a line coming fro*m* the opposite angle. the triangle is devided into two a*e*quall partes. so likewise in a sector of a circle.



Note whether it be so in the sector of a helicall & what proportion the angles become.

The longitudes of euery helicall is geven & therefore any proportionall parte, as namely 1/10. The quantity also of euery superficies included by the helicall & therefore any proportionall parte as namely 1/10.

Therefore a first helicall triangle is geven in superficies & the first side *ab*. & the helicall base, & the angle conteyned by the first side *ab* & the helicall base *bf*. Then why shold not *af* be knowne the other side.

If euery helicall be a*e*quall to the secant of his angle why shold not any segment of a helicall be the secant also of his angle. And is it less: the tangent ⟨wilbe⟩ of the angle will not be the perpendicular drawne fro*m* the vpper end of the segment to the meridian fro*m* whence the other end of the segment doth rise; but that perpendicle that is vnderstoode to fall to the sayd meridian from the vpper end of the segment streched as strayte in the same angle.

A line drawne fro*m* the poynt streched out & the poynt of the helicall as it was naturall what angle it will make being drawne out with the sayd meridian.

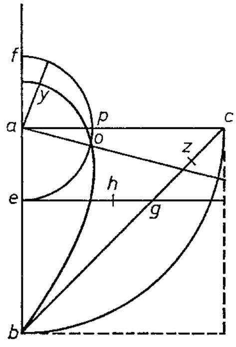
To get the last proportionall of those that instyt vpon the helicall of $4\bar{5}$. that is to say that which with the first maketh $9\bar{0}$ angle. *ab*, being the first, *af*, the second, then *ag* & so forth till *az* the desired. The angle *baf* being one minute. 5401 is the nomber of places in the progression, *ab* being the first & *az* last.

In the first charte of Helicall, if *bf* be the segment of a helicke streched out & *kf* the parallel streched out. *bk* wilbe knowne & *kf*. so will the parallel described fro*m* *k*, & what? parte *Hf* is of the parallel, that wilbe the angle of *ba* & *af* in ... (?), vpon the helicall in his owne nature.

Whether triangles may have easier solutions by helickes or not.

The area of a triangle being geuen & the rate of 2 sides, all the sides are geuen.

(f. 241) *b*.2) De helicis



′	″	‴	⁗
ba.	*bc.*	*be.*	*bg.*
100,000|0	141421|3	73,205.	103,527
ba	*ao* = *ae*	*bo* = *bg*	*oy* = *gz*
100,000|0	26,794,|9	103,527	27,739,|9

′ ″

bo. *oy.* &c. = *bc*:

bo
bo = *bc*.
bo — *by*
bo = 10,717,839,729 0
bo

bo.	103,527,|4
— *oy*	27,739,|9
	75,787 |5

[There now follows the division of bo^2 by $bo - oy$ to give:]

14 141 96. = *bc*. satis recte.
Nil tamen concludit.

(f. 269. d. longs. on the 7th rhumb)*

0. 0′	0. 00′, 00
1. 0′	5. 01′, 66
2. 0′	10. 03′, 40
3. 0′	15. 05′, 33
4. 0′	20. 07′, 54
5. 0′	25. 10′, 12
. . . .	. . . . . .
57. 0′	350. 27′, 45
58. 0′	359. 48′, 89
59. 0′	1° 9. 26′, 24
60. 0′	1° 19. 20′, 61
. . . .	. . . . . .
80. 0′	1° 341. 44′, 96
81. 0′	2° 12. 14′, 24
. . . .	. . . . . .
88. 0′	3° 86. 02′, 62
89. 0′	3° 285. 43′, 40
89.59′	7° 25. 05′, 25

* It is to be remembered that the notation 1° refers to one complete turn of the globe, that is, 360 degrees.

(f. 272. The γ-tables)

0′,00	10 000 000 000 0	
	29 088 7 787	29 088 7 787
01	9 999 970 911 2 213	0 846
	29 088 6 941	29 088 6 941
02	9 999 941 822 5 272	846
	29 088 6 095	29 088 6 095
03	9 999 912 733 9 177	846
	29 088 5 249	29 088 5 249
04	9 999 883 645 3 928	846
	29 088 4 403	29 088 4 403
05	9 999 854 556 9 525	846
. . .	. . . .	. . . . .
10	9 999 709 116 0 200	846
	29 087 9 327	29 087 9 327
. . .	. . . .	. . . . .
19	9 999 447 327 6 713	846
	29 087 1 713	29 087 1 713
20	9 999 418 240 5 000	846

(f. 279) *a.c.*3.) Qua*e*ritur terminus 20us vnde $n = 20$. nn $= 400$.

1°.)

p.	29,088,7,787
e	0,846
$p+e$	29,088,8,633
$-en$	$-$ 1,6,920
$p+e-en$	29,087,1,713 $=$ v′ $=$ 20.

2°.)

$2p^2$	19,999,941,822,4,426
$+2p$	58,177,5,574
$+enn$	33,8,400
	20,000,000,033,8,400
$-2pn-en$	1,163,552,8,400
	19,998,836,481,0,000. $= 2,v^2$
	9,999,418,240,5,000. $= v^2 = 20$.

[&c.]

(f. 281) *a.c.*1)

	10,002,909,305,1	
	2,909,305,1	2,909,305,1
0′	10,000,000,000,0	846,0
	2,908,459,1	2,908,459,1
1′	9,997,091,540,9	846,0
	2,907,613,1	2,907,613,1
2′	9,994,183,927,8	

Canon.)

$$\begin{array}{c|c} \bigtriangledown & \square \\ \dfrac{2pn + en - enn}{2} & enn \end{array} \qquad \text{species totius progress}\textit{ionis} \quad \bigtriangledown.\ \bigtriangledown.\ \square.$$

sit progressio dimidienda in 100, partes.

ergo: $n = 100.$

$$enn = 10{,}000, \qquad e = 8460$$

$$e = \frac{8460}{10000} = \frac{846}{1000}$$

$$\frac{2pn + en - enn}{2} = 2{,}908{,}459{,}1$$

ergo $2pn + en - enn = 5{,}816{,}918{,}2$

hoc est: $200{,}p + \underbrace{\frac{846\ 000}{10000}}_{= + \frac{846}{10}} - 8460 = 5{,}816{,}918{,}2$

Ergo $200{,}0{,}p + 846 - 8460{,}0 = 5{,}816{,}918{,}2{,}0$

$$2000{,}p + 846 = 5{,}816{,}918{,}20$$

$$+ 846\,00$$

$$2000{,}p = \begin{cases} 5{,}817{,}764\,20 \\ - \quad 8\,46 \end{cases}$$

$$2000{,}p = 5{,}817{,}755{,}74$$

$$p = \frac{5{,}817{,}755{,}74}{2000}$$

$$p = 29{,}088{,}7{,}\frac{787.}{1000}$$

(f. 297 *et seq*. The β-tables)

1′	10 002 909 305 186 9		
		2 909 305 186 7	
0′	10 000 000 000 000 0		846 159.5
		2 908 459 027 4	246 1
1′	9 997 091 540 972 5		845 913.4
		2 907 613 114 0	246 1
2′	9 994 183 927 858 5		845 667 3
		2 906 767 446 7	245 8

. . . .

(f. 298)

11′	9 968 053 435 363 4		843 456 3
		2 899 167 500 0	245 3

. . . .

(f. 299)

51′	9 852 742 023 424 8 2 865 629 648 3	833 699 1 2 865 629 648 3	833 699 1 242 5 833 456 6
. . .	. . .	. . .	. . .

(f. 300)

60′	9 826 981 340 675 9 2 858 137 259 2	831 519 4 2 858 137 259 2	831 519 4 241 9
61′	9 824 123 203 416 7 2 857 305 981 7	831 277 5 2 857 305 981 7	831 277 5 241 8 831 035 7

(ff. 301—303. The α-tables)

$\bar{0}$	10 000 000 000 000
$\bar{1}$ $\bar{2}$	9 826 981 340 675 9 656 956 226 999
$\bar{3}$ $\bar{4}$	9 489 872 865 044 9 325 680 357 018
$\bar{5}$ $\bar{6}$	9 164 328 685 752 9 005 768 699 471
$\bar{7}$ $\bar{8}$	8 849 952 096 814 8 696 831 412 127
$\bar{9}$ $1\bar{0}$	8 546 360 000 997 8 398 492 026 050
. . .	. . .
$2\bar{0}$	7 053 466 831 163

(f. 303)

$13\bar{0}$	1 034 229 889 245
$13\bar{1}$ $13\bar{4}$	1 016 335 782 358 964 489 736 277 *a*
$13\bar{9}$ $14\bar{5}$	883 890 095 728 *a* 796 010 975 788 5
$15\bar{2}$ $16\bar{0}$	704 465 900 426 7 612 662 117 160 4
. . .	. . .
$23\bar{1}$ $27\bar{1}$	177 439 521 745 6 88 278 636 175 71
$50\bar{6}$	1 460 785 667 132

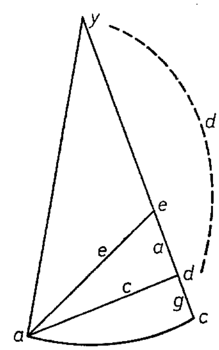
(f. 302)

. . .	. . .
$9\bar{8}$	1 807 893 649 570
$9\bar{9}$ $10\bar{0}$	1 776 613 716 024 1 745 874 983 697

(f. 432) $\surd,cc + 2gg - 2,g = a$

vel: $\surd,3bb + dd - 4bd, - 2g = a$

$b.$	10,000,000,000.
$3bb.$	30,000,000,000,000,000,000,0.
$dd.$	9,999,999.153,840,524,458,7
$3bb + dd.$	39,999,999,153,840,524,458,7
$-4bd.$	39,999,998,307,681,013,118,0
$3bb + dd - 4bd$	846,159,511,340,7
$\surd,3bb + dd - 4bd$	2,908,882,*107,168,8*
$b.$	10,000,000,000
$d.$	9,999,999,576,*920,253,279,5*
$b - d. = g$	423,*079,746,720,4*
$2g$	846,*159,493,440,8*
$\surd,\underbrace{3bb + dd - 4bd - 2g}_{a}$	2,908,035,*947,675,4aa,a*
$\underbrace{d - a}_{ye}$	9,997,091,540,*972,577,8aa,a*



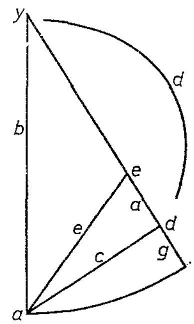
$aa.$	845,667,307,297,2
$cc.$	846,159,475,541,2
$cc + aa$	1,691,826,782,838,4
$\surd,cc + aa = e = ae$	4,113,182,*202,186,5aa,a*
$ay - ye$	2,908,459,*027,422,1aa,a*
Ergo Helix ex rectis *ae*, &c	14,142,135,623,*729*
Helix vera *f*.	14,142,135,623,*730,950,488,0*

$e = ae$	4,113,182,*202,186,5aa,a*
Helix vera $-$ *ae*	14,138,022,441,*528,763,9,aaa*

$\overset{\prime}{f}$ $\quad \overset{\prime\prime}{f - e}$ $\quad \overset{\prime\prime\prime}{ay}$ $\quad \overset{\prime\prime\prime\prime}{ye}$

ye vt *ae* posita curva } 9,997,091,540,*972,578,3* non 4. vel. 5.

(f. 434)



$$\frac{be}{b-d+a} = f$$

$$be = bf - df + fa$$

$$e = \frac{bf - df + fa}{b}$$

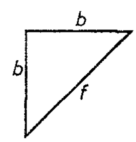
$$e = \sqrt{,cc + aa}.$$

$$\sqrt{cc + aa} = \frac{bf - df + fa}{b}$$

$$\text{sit, } g = b - d$$

$$\sqrt{cc + aa} = \frac{gf + fa}{b}$$

$$cc + aa = \frac{ggff + 2gffa + ffaa}{bb}$$



$$bbcc + bbaa = ggff + 2gffa + ffaa$$

$$ff = 2bb$$

$$\cancel{b}\cancel{b}cc + \cancel{b}\cancel{b}aa = 2gg\cancel{b}\cancel{b} + 4g\cancel{b}\cancel{b}a + 2\cancel{b}\cancel{b}aa^{\star}$$

$$cc - 2gg = 4ga + aa$$

$$+4gg+$$

$$cc + 2gg = 4gg + 4ga + aa$$

$$\sqrt{cc + 2gg} = 2g + a$$

$$\sqrt{cc + 2gg} - 2g = a.$$

$$b - d = g.$$

$$bb - 2bd + dd = gg$$

$$2bb - 4bd + 2dd = 2gg.$$

$$bb - dd = cc$$

$$\sqrt{bb - dd + 2bb - 4bd + 2dd} - 2g = a.$$

$$\sqrt{,3bb + dd - 4bd} - 2g = a.$$

---

* The $\cancel{b}$ here is printed for a cancelled b in the original.

(BM Add MS 6782, f. 67) Interest vpon interest for 7 yeares

$$\begin{array}{c|l}
\frac{7}{1} & bb + \frac{7b}{1} + \frac{21}{1} + \frac{35}{b} + \frac{35'}{bb} + \frac{21}{bbb} + \frac{7}{bbbb} + \frac{1}{bbbbb} \\[2ex]
\frac{14}{2} & bb + \frac{14b}{2} + \frac{91}{4} + \frac{364}{8,b} + \frac{1001}{16,bb} + \frac{2002}{32,bbb} \quad \&c. \\[2ex]
\frac{21}{3} & bb + \frac{21b}{3} + \frac{210}{9} + \frac{1330}{27,b} + \frac{5985}{81,bb} + \frac{20349}{143,bbb} \quad \&c. \\
\hline
\end{array}$$

$$\begin{array}{llr|llll}
bb + & \frac{7b,n}{1n} & +7 & n-1+ & 7n-2\ + & 7n-3\ + & 7n-4 \quad \&c. \\
 & & 7 & n & 7n-1 & 7n-2 & 7n-3 \\
 & & 2 & n & 7 & 7n-1 & 7n-2 \\
 & & & n & n & 7n & 7n-1 \\
 & & & & n & n & 7n \\
 & & & & n & n & n \\
 & & & & 6,b & n & n \\
 & & & & & n & n \\
 & & & & & 24,bb & n \\
 & & & & & & n \\
 & & & & & & 120,bb
\end{array}$$

$$100 + \frac{70}{1} + \frac{49}{2} + \frac{343}{6,0} + \frac{2401}{24,00} + \frac{16,807}{120,000} + \frac{197,649}{720,000} + \frac{823,543}{5040,00000} +$$

$$+ \frac{5764801}{40320,000000}. \quad \&c.$$

[Each term is now worked out in pounds, shillings and pence, and the total found, which is:]

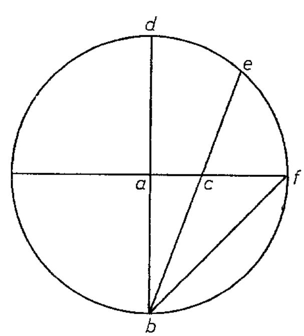
$$201 + 7 + 6\frac{06205a}{1000000} \qquad \text{not } \frac{7}{100}.$$

The summe of interest vpon interest continually for euery instant in seven yeares with the principall of 100 *l* after the rate of 10 in the 100 for a yeare.

(BM Add MS 6786, f. 87) De logarithmis

ba.	10,000,000,*000*
♈.0̄.1′	2,908,*882*
log:	81,425,681
♈.0̄.2′.	5,817,*764*
log:	74,494,213

1.)



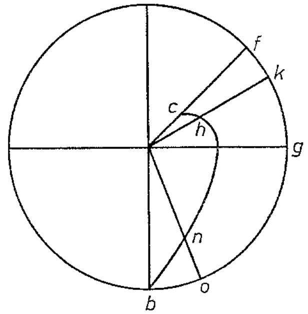
sit *ac*. 2908,*882*. ♉.1′ 2.)

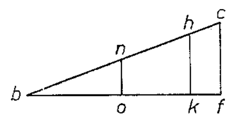
est etiam ♉.1′
ergo: *de*. 2′.
et vt *af* 9$\bar{0}$ erit: *ac*. 2′. et *cf*: 8$\bar{9}$. 58′.
Nautice: 8$\bar{9}$. 58′.) $\underbrace{46\bar{6},32'}_{27,992'.}$,099,*074*.

bf. 81,425,681. *fc*. 27,992,*099*

10,000,000. 3,437,*748,2*

2,908,880,*87a,a* 10,000,000.





(BM Add MS 6789, f. 17 Stereographic projection.)

Quod rumbus in planisp*hae*rio nostro facit angulum cu*m* meridiano a*e*qualem angulo facto a rumbo cu*m* meridiano in spha*e*ra.

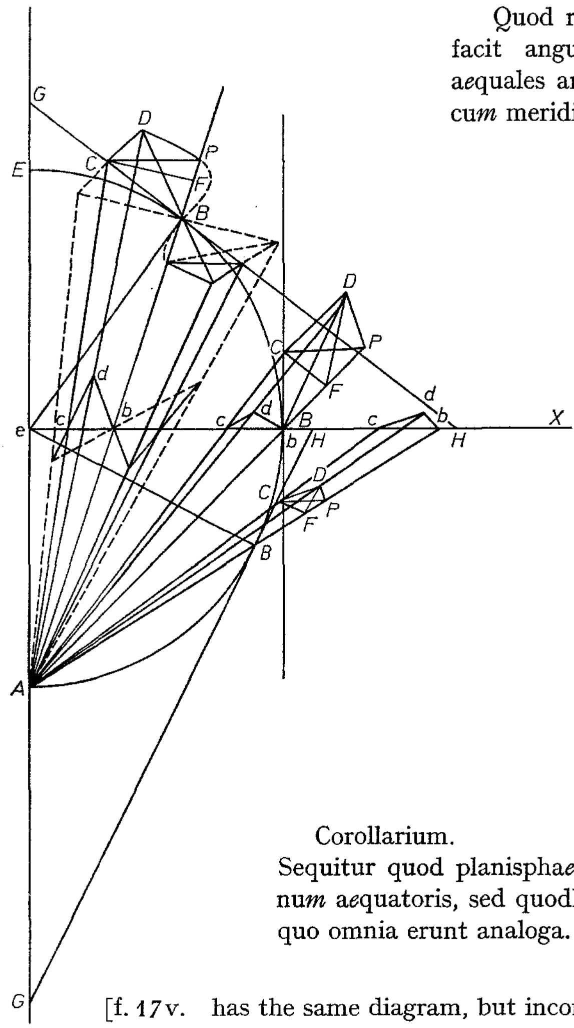
Sit *ABE* semicirculus pro meridiano quolibet in globo terrestri. Cuius centru*m* *&*. poli *A*. et *E*. ⟨In illo meridiano sumatur quoduis punctu*m* *B*⟩. Sit, *&X*, communis sectio a*e*quatoris et plani meridiani qua*e* meridianus est in nostro planisphaerio correlatiuus meridiano in globo terrestri. In semicirculo pro meridiano sumatur quoduis punctu*m* *B*. Agatur recta *AB* producta qua*e* secabit, *&X*, in puncto *b*, et punctu*m* *b* in planisphaerio erit correlatiuu*m* puncto *B* in superficie spha*e*ra*e*. Agatur etiam recta, *&B*, et *GH* tangens peripheriam in puncto, *B*. Dico primo q*uod* anguli *HBb* et *HbB* sunt a*e*quales. *HBb*+*AB&* ⫨ recto. *&bA* (⫨ *HbB*) + *&AB* ⫨ recto. Ergo: *HBb* + *AB&* ⫨ *&bA* (⫨ *HbB*) + *&AB*. Sed: *AB&* ⫨ *&AB*. ergo: *HBb* ⫨ *HbB*. Iam intelligitur *B* punctu*m* esse centru*m* circuli nautici qui vulgo compassus dicitur; sive circuli horizontis visibilis. Cuius plano, *&B*, est ad angulos rectos. *GH* est com*m*unis sectio plani meridiani et illius circuli nautici, et est linea dicta meridiana in plano horizontis ⟨circuli nautici⟩. Agatur quaelibet linea *BD* in plano horizontis faciens ⟨quemuis⟩ angulu*m* *DBG* ⟨cum meridiana *BG*⟩ et in illa linea sumatur quoduis punctu*m* *D* et Agatur *DC* perpendicularis ad *GH*. Con*n*ectantur puncta *AC*: et *AD*. et constituatur pyramis cuius basis *BCD*, et vertex, *A*. Quoniam *CD*, perpendicularis est in plano erecto ad planu*m* meridianis circuli productu*m* erit etiam perpendicularis ad *CA*, et planu*m* trianguli *DCA* erit erectu*m* ad planu*m* meridiani. Linea *AC*, secat, *&X*, in puncto *c* (non opus est ducere lineas *DF*, et *CF*. indeo *que* omittuntur). Et in plano *ACD* a puncto *c* in linea *AC* erigatur, *cd*, ad angulos rectos, quae secabit *AD* in puncto *d*. Et erit com*m*unis sectio a*e*quatoris et trianguli *ACD*. Con*n*ectantur puncta *d*, *b*. Dico q*uod* angulus *dbc* in plano a*e*quatoris est a*e*qualis angulo *DBC* in plano horizontis sive circuli nautici. producatur *AB* et sit *CF* perpendicularis ad illam. fiat *FP* ⫨ *FB* et agantur *FC*, *FD*, *PC*, et *PD*. Quoniam *DC* est perpendicularis ad planu*m* meridiani, facit rectos angulos

cu*m* CA, CB, CF, et CP. Cu*m* etiam CB et CP, sunt a*e*quales triangula rectangula BCD et PCD, sunt a*e*qualia et a*e*quiangula, et angulus CBD a*e*qualis est CPD. Sed angulus cbd in a*e*quatore est a*e*qualis CPD, ob parallelismum triangulorum cbd et CPD nam cd est parallela CD et cb, CP, quia: $cbA = CPA = CBP = HBb = HbB$ ergo tertiu*m* latus db est parallelu*m* DP ergo plana triangulorum cbd et CPD sunt parallela. Et similia. et angulus $cbd = CPD$. Ergo $cbd = CBD$. quod demonstrandu*m* fuit. Aliter db est parall. DP, quia

$$\overset{\prime}{AP},\quad \overset{\prime\prime}{Ab}:\quad \overset{\prime\prime\prime}{AC},\quad \overset{\prime\prime\prime\prime}{Ac}$$

$$AD \qquad Ad$$

(BM Add MS 6789, f. 18)



Quod rumbus in planisp*h*a*e*rio nostro facit angulos cu*m* meridiano quolibet a*e*quales angulis factis a rumbo correlatio cu*m* meridiano a sp*h*a*e*ra.

correspondens*
correlatiuum
proiectu*m*
coniugatum
cognomine
cognatu*m*
representatiuu*m*
perspectiuu*m*

Corollarium.

Sequitur quod planispha*e*riu*m* non erit solum*m*odo planu*m* a*e*quatoris, sed quodlibet planum illi parallelum, in quo omnia erunt analoga.

[f. 17v. has the same diagram, but incomplete.]

* These may be alternatives for the partially cancelled word *correlatio* of the theorem.

English Version of the Proof of Stereographic Projection

In our planisphere a rumb makes with any meridian angles equal to the angles made by the corresponding rumb with the meridian on the sphere.

Let ABE be a semicircular meridian section of the terrestrial globe, whose centre is $\mathscr{E}$, poles A and E. Let $\mathscr{E}X$ be the intersection of the meridian and equatorial planes, which meridian corresponds to that of our planisphere. Consider a point B on the semicircle. Produce the line AB to meet $\mathscr{E}X$ in the point b, and then the point b in the planisphere corresponds to the point B on the sphere. Now draw the line $\mathscr{E}B$, and the tangent GH to the circle at B. I say first that $H\hat{B}b = H\hat{b}B$, because $H\hat{B}b + A\hat{B}\mathscr{E} = 90°$, and $\mathscr{E}\hat{b}A$ $(= H\hat{b}B) + \mathscr{E}\hat{A}B = 90°$. Hence $H\hat{B}b + A\hat{B}\mathscr{E} = H\hat{b}B + \mathscr{E}\hat{A}B$. But $A\hat{B}\mathscr{E} = \mathscr{E}\hat{A}B$, so that $H\hat{B}b = H\hat{b}B$. Now it is understood that the point B is the centre of the nautical circle (or compass card), or circle of the visible horizon, which is perpendicular to the plane of EB. GH is the line common to the meridian plane and that of the nautical circle, and is called the meridian line of the horizontal plane. Draw any line BD in the plane of the horizon, making any angle $D\hat{B}G$ (with the meridian line BG) and on that line choose any point D and draw DC perpendicular to GH. Join the points AC, and AD, to form a pyramid whose base is BCD and vertex A. Since CD is perpendicular to the plane of the meridian, it is also perpendicular to AC, and the plane of the triangle DCA will be at right-angles to the plane of the meridian. Let the line AC cut $\mathscr{E}X$ in a point c.

Now draw cd, in the plane ACD, perpendicular to AC, so as to cut AD in the point d. It will then be the line common to the plane ACD and the equatorial plane. Join the points d and b. I say that the angle $d\hat{b}c$ in the equatorial plane is equal to the angle $D\hat{B}C$ in the plane of the horizon, or nautical circle. Draw CF perpendicular to AB produced. Make $FP = FB$ and draw FC, FD, PC, PD. Since DC is perpendicular to the plane of the meridian, it is also perpendicular to CA, CB, CF, and CP. Since also CB and CP are equal, the right triangles BCD and PCD are equal and similar, and the angle $C\hat{B}D$ is equal to $C\hat{P}D$. But the angle $c\hat{b}d$ in the equatorial plane equals $C\hat{P}D$, because the triangles cbd and CBD are parallel. For cd is parallel to CD, and so are cb and CP, because $c\hat{b}A = C\hat{P}A = C\hat{P}B = H\hat{B}b = H\hat{b}B$. Hence the third side db is parallel to DP, and so the planes of the triangles cbd and CPD are parallel. Similarly, the angles $c\hat{b}d = C\hat{P}D$. Hence $c\hat{b}d = C\hat{B}D$, which was to be proved. (Alternatively, db is parallel to DP, because $AP/Ab = AC/Ac = AD/Ad$.)

Corollary. It follows that the planisphere can be not merely the equatorial plane, but any plane parallel to it, in which case the argument would be wholly analogous.

Mathematics Department
Royal Naval College
Greenwich, England

(Received December 6, 1967)